

# The sector constants of continuous state branching processes with immigration <sup>1</sup>

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*Dedicated to Professor Tadahisa Funaki on the occasion of his 60th birthday*

Continuous state branching processes with immigration are studied. We are particularly concerned with the associated (non-symmetric) Dirichlet form. After observing that gamma distributions are only reversible distributions for this class of models, we prove that every generalized gamma convolution is a stationary distribution of the process with suitably chosen branching mechanism and with continuous immigration. For such non-reversible processes, the strong sector condition is discussed in terms of a characteristic called the Thorin measure. In addition, some connections with notion from noncommutative probability theory will be pointed out through calculations involving the Stieltjes transform.

## 1 Introduction

Besides its significance in the physical context, the (time-)reversibility can be thought of as a mathematical condition which guarantees a certain kind of ‘solvability’ of the equilibrium state and usually makes one possible to deduce explicit consequences. On the other hand, it is likely that the reversibility is a restrictive condition, and it fails for a number of stochastic models with stationary distributions of interest. In this paper, our attempt will be made in quantitative discussions on the degree of irreversibility of such systems. Let us illustrate roughly in a general setting the situation we will be concerned with. Suppose that we are given a Markov process with a stationary distribution  $\nu$  and generator  $L$ , say. Then consider a bilinear form  $\mathcal{E}$  defined by

$$\mathcal{E}(f, g) = - \int Lf(x)g(x)\nu(dx), \quad f, g \in D(L), \quad (1.1)$$

where  $D(L)$  is the domain of  $L$ . The Dirichlet form is a suitable extension of  $\mathcal{E}$  and it is well-known that the symmetry of  $\mathcal{E}$  is interpreted as the reversibility of the Markov

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process. We say that  $\mathcal{E}$  satisfies the strong sector condition if there exists a finite constant  $C$  such that

$$\mathcal{E}(f, g) \leq C \mathcal{E}(f, f)^{1/2} \mathcal{E}(g, g)^{1/2}, \quad f, g \in D(L). \quad (1.2)$$

Based on a weaker version called the weak sector condition, the theory of symmetric Dirichlet forms has been successfully extended to non-symmetric cases in [12]. The strong sector condition is known also to play an essential role in the proof of the invariance principle for additive functionals of non-symmetric Markov processes. See [15] and [23]. (See also [9] for a generalization.) Intuitively, the validity of this condition tells us that the process is a small perturbation from a symmetric one. It seems typical that verification of (1.2) depends heavily on the mathematical structure of the process. In our subsequent discussions it will be convenient to denote by  $\text{Sect}(\mathcal{E})$  the infimum of  $C$ 's satisfying (1.2) if any, and set  $\text{Sect}(\mathcal{E}) = \infty$  otherwise. We call  $\text{Sect}(\mathcal{E})$  the sector constant of  $\mathcal{E}$ . Clearly  $\text{Sect}(\mathcal{E}) \geq 1$ . If  $\mathcal{E}$  is symmetric, we have  $\text{Sect}(\mathcal{E}) = 1$ . As expected, the converse holds true in general. (See Proposition 3.1 below for the proof.) Hence the difference  $\text{Sect}(\mathcal{E}) - 1$  can be thought of as a 'degree' of asymmetry of  $\mathcal{E}$  and of irreversibility of the process. Our objective is to show such a property of  $\text{Sect}(\mathcal{E})$  in an explicit way for some specific class of models. We thus seek for an upper bound of  $\text{Sect}(\mathcal{E}) - 1$  which should be given in terms of certain characteristics of the models, and the bound is then required to vanish precisely in the reversible case.

As a model which will be discussed in the present paper, we adopt the continuous state branching process with immigration (called also the CBI-process), which is a Markov process on  $\mathbf{R}_+ := [0, \infty)$ . Fundamental results of this process, including limit theorems from Galton-Watson processes with immigration and the complete determination of the generator, are obtained by Kawazu and Watanabe [7]. Since then, this model has been studied extensively not only because of the rich and nice mathematical structure which has allowed us to obtain a number of concrete results of interest but also of its importance in various applications. The aforementioned authors showed its interesting applications in the context of stochastic analysis as well. In addition, since the CIR model [1], a mathematical finance model for evolution of interest rate, is included as a special case (in fact, the diffusion case), the class of CBI-processes serves also as a useful generalization of the CIR model in such a context [4]. We intend to reveal further aspects of the CBI-process regarding the non-reversible stationary distribution and the non-symmetric Dirichlet form.

The time evolution of the CBI-process in general incorporates two kinds of dynamics; the one describing the branching of particles and the other being due to immigration. It is of essential importance to take into consideration the effect of immigration. One of consequences of introducing immigration is ergodicity of the process; it may exhibit the strong convergence to a unique stationary distribution, if any, as time goes to infinity. Actually, under suitable assumptions, the positivity of the spectral gap of  $L$  follows from the result in [14]. (See the discussion in the paragraph preceding to Lemma 2.1 below for the precise statement.) Furthermore, our process has so nice structure as to make it possible to get information of the

stationary distribution through an explicit representation of the Laplace transform. This formula, a key tool throughout this paper, is due to Ogura [14], who carried out detailed calculations of the spectral representation for the CBI-process. However, it is typically difficult to deduce direct expressions (e.g. the density function) of the stationary distribution and so one needs to exploit other structures. (Among exceptions are gamma distributions, which are reversible distributions of the CIR models.) Another feature of the model which is crucial to us is the branching property, meaning that the law of the sum of two identical and independent processes starting from  $x_1$  and  $x_2$  respectively coincides with the law of a process starting from  $x_1 + x_2$ . By virtue of this property, the law of the process at an arbitrarily fixed time is necessarily infinitely divisible and so is the stationary distribution. (See [8] for recent studies of stationary distributions of the CBI-processes.) In [21], the Poincaré inequality for a class of infinitely divisible distributions on  $\mathbf{R}_+$  (and more general spaces) was proved by reducing it to an analogous estimate for the associated Lévy measure. It will turn out that a suitably modified argument works well for the sector constant estimate.

Non-reversible stationary distributions we will focus attention on are generalized gamma convolutions [2] (GGC's for short), namely weak limits of finite convolutions of gamma distributions. (See also Section 5, Chapter VI of [22] for general accounts and [6] for a recent survey and related topics.) We regard these distributions as 'perturbations' from gamma distributions. A GGC without 'translation term' is determined uniquely by the so-called Thorin measure, which appears in the logarithm of the Laplace transform and prescribes the weight of convolutions. For example, every gamma distribution has a degenerate Thorin measure. Therefore, the actual problems we are going to consider in the subsequent sections are outlined as follows.

- (I) Show that there does not exist a (nondegenerate) reversible distribution of the CBI-processes except gamma distributions.
- (II) Given a GGC with Thorin measure  $m$ , choose a branching mechanism so that the CBI-process has the GGC as a unique stationary distribution.
- (III) For the bilinear form  $\mathcal{E}$  associated with that process, give an upper bound  $C = C(m)$  of  $\text{Sect}(\mathcal{E})$  such that  $C(m) = 1$  if and only if  $m$  is degenerate.

We will see that the reversibility problem (I) reduces to solving certain functional equations involving 'characteristics' of the mechanisms of branching and immigration. Our solution to (II) will turn out to rely on the theory of Bernstein functions [18]. We also make use of Stieltjes transforms in order to get further information (e.g., the one needed to solve (III)) on the branching mechanism chosen. In this context some connections with notion from non-commutative probability theory (such as the so-called Boolean convolution and the free Poisson distribution) will be pointed out.

The organization of this paper is as follows. In the next section a precise description of CBI-processes is given and then the problem (I) is solved. In Section 3, we present some basic results on the strong sector condition for a subclass of CBI-processes for which an integration by parts formula is available. In Section 4, both the problems (II) and (III) are solved by constructing the CBI-process associated with a GGC and then applying the results in Section 3. In Section 5, we give some examples to illustrate consequences of our results and discuss related topics.

## 2 The model and its stationary distribution

Following [7], we begin with a precise description of our model, namely the CBI-process in terms of the generator. For the purpose of this paper, we shall restrict the discussion to a class of conservative CBI-processes. In view of Theorem 1.1', Theorem 1.2 in [7], and results (especially Proposition 1.1) in [14], the assumptions made below are not optimal but useful in order that the results of this section are not more complicated than are necessary in the subsequent sections. (Recently, the detailed analysis of stationary distributions was done in [8] for conservative CBI-processes.) The generator  $L$  of our process takes the following form:

$$\begin{aligned} Lf(x) = & axf''(x) - bxf'(x) + x \int_0^\infty [f(x+y) - f(x) - yf'(x)] n_1(dy) \\ & + \delta f'(x) + \int_0^\infty [f(x+y) - f(x)] n_2(dy), \quad x \in \mathbf{R}_+, \end{aligned} \quad (2.1)$$

where  $a \geq 0, b \geq 0, \delta \geq 0$ , and measures  $n_1$  and  $n_2$  on  $(0, \infty)$  are supposed to satisfy

$$\int_0^\infty \min\{y^2, y\} n_1(dy) + \int_{(0,1)} y n_2(dy) + \int_{[1,\infty)} (1 + \log y) n_2(dy) < \infty. \quad (2.2)$$

This process approximates (asymptotically critical) Galton-Watson branching processes with immigration in large population limit. In this context dynamical meaning of the constants and measures appearing in (2.1) may be explained as follows. While  $a$  is the asymptotic variance of the offspring distributions associated with the branching mechanisms,  $b$  comes from the first order approximation to mean 1.  $\delta$  is the rate of change in mean of immigrating population.  $n_1$  and  $n_2$  describe effects of big changes in population size which occur in 'macroscopic time scale' and are caused by branch(-death) and immigration, respectively. To avoid the triviality in discussing the equilibrium of the model, we make the assumption implying that both branching and immigration mechanisms are actually present. To be precise, defining for  $\lambda \geq 0$

$$R(\lambda) = -a\lambda^2 - b\lambda - \int_0^\infty (e^{-\lambda y} - 1 + \lambda y) n_1(dy)$$

and

$$F(\lambda) = \delta\lambda + \int_0^\infty (1 - e^{-\lambda y}) n_2(dy),$$

we assume throughout that

$$\text{neither } R \equiv 0 \text{ nor } F \equiv 0. \quad (2.3)$$

The functions  $R$  and  $F$  are called the branching mechanism and the immigration mechanism, respectively, and their interplay will be crucial in the ergodic behavior of the CBI-process.

As in the literature on CBI-processes, a large amount of calculations below will be based on the Laplace transforms, which can be expressed in terms of the associated

$\Psi$ -semigroup, a one-parameter family  $\{\psi(t, \cdot)\}_{t \geq 0}$  of non-negative functions on  $\mathbf{R}_+$  determined by the equation

$$\frac{\partial \psi}{\partial t}(t, \lambda) = R(\psi(t, \lambda)), \quad \psi(0, \lambda) = \lambda \quad (2.4)$$

with  $\lambda \geq 0$  being arbitrary. Let  $T_t$  be the semigroup of the CBI-process, and for every  $\lambda \geq 0$  define a function  $f_\lambda$  on  $\mathbf{R}_+$  by  $f_\lambda(x) = e^{-\lambda x}$ . Then by Theorem 1.1 in [7]

$$T_t f_\lambda(x) = \exp\left(-x\psi(t, \lambda) - \int_0^t F(\psi(s, \lambda))ds\right), \quad t, \lambda \geq 0. \quad (2.5)$$

Ogura's formula (Eq.(1.12) with  $\alpha = 0$  in [14]) for a unique stationary distribution, say  $\nu$ , of this process is

$$\int_{\mathbf{R}_+} f_\lambda(x) \nu(dx) = \exp(-\Phi(\lambda)), \quad \lambda \geq 0, \quad (2.6)$$

provided the 'Laplace exponent'  $\Phi$  given by

$$\Phi(\lambda) = - \int_0^\lambda \frac{F(u)}{R(u)} du \quad (2.7)$$

is finite for all  $\lambda > 0$ . Conversely, if the CBI-process has a stationary distribution, then  $\Phi(\lambda) < \infty$  for all  $\lambda > 0$  and (2.6) holds. (See Lemma 2.1 below for the proof.)

As shown in [14], the constant  $b$  plays an important role in studying ergodic properties of the process. For instance, under the assumptions  $b > 0$  and that both  $R$  and  $F$  are analytic at  $\lambda = 0$ , the spectral representation of Theorem 3.1 in [14] implies in particular that  $0, b, 2b, \dots$  form the discrete spectrum of  $-L$ . Assuming the finiteness of  $\Phi$  only, we will see below the convergence of the transition function as  $t \rightarrow \infty$ . In such a case, the stationary distribution  $\nu$  is necessarily infinitely divisible for the reason mentioned in Introduction, and therefore  $\Phi$  is expressed uniquely in the form

$$\Phi(\lambda) = q\lambda + \int_0^\infty (1 - e^{-\lambda y}) \Lambda(dy) \quad (2.8)$$

for some  $q \geq 0$  (the 'translation term') and measure  $\Lambda$  (called the Lévy measure) on  $(0, \infty)$  such that  $\int_0^\infty \min\{1, y\} \Lambda(dy) < \infty$ . (See e.g. §51 of [17].) Obviously  $q$  is interpreted as the infimum of the support of  $\nu$ . The condition that  $\Phi(1) < \infty$  is sufficient to guarantee that  $\Phi(\lambda) < \infty$  for every  $\lambda > 0$  since by integration by parts

$$R(\lambda) = -a\lambda^2 - b\lambda - \lambda \int_0^\infty (1 - e^{-\lambda y}) \widetilde{n}_1(dy) < 0, \quad \lambda > 0, \quad (2.9)$$

where  $\widetilde{n}_1(dy) = n_1([y, \infty))dy$ . Incidentally, we remark that  $(0, \infty) \ni \lambda \mapsto -R(\lambda)/\lambda$  defines a Bernstein function with characteristic triplet  $(b, a, \widetilde{n}_1)$  in the terminology of [18] (Chap.3, Theorem 3.2). Let  $P_t(x, dy)$  denote the transition function of the CBI-process. The following lemma gives basic observations concerning ergodicity and can be deduced from the results announced in [16]. The proof was given in [11]. (See Theorem 3.20 and Corollary 3.21 there.)

**Lemma 2.1** (i) Assume that  $\Phi(1) < \infty$  and let  $\nu$  satisfy (2.6). Then, for each  $x \in \mathbf{R}_+$ ,  $P_t(x, \cdot) \rightarrow \nu$  weakly as  $t \rightarrow \infty$ .

(ii) Suppose that the CBI-process has a stationary distribution, then  $\Phi(1) < \infty$ .

(iii) If  $b > 0$ , then  $\Phi(1) < \infty$ .

In our discussion, a CBI-process is said to be ergodic if it has a (unique) stationary distribution, or equivalently  $\Phi(1) < \infty$ . The next proposition concerns not only the translation term  $q$  of the stationary distribution in the ergodic case but also the infimum, denoted by  $q(t, x)$ , of the support of  $P_t(x, \cdot)$ . Put  $c = \int_0^\infty \widetilde{n}_1(dy) = \int_0^\infty y n_1(dy)$ .

**Proposition 2.2** (i) If  $a > 0$  or  $c = \infty$ , then  $q(t, x) = 0$  for any  $x \in \mathbf{R}_+$  and  $t > 0$ . Under the additional condition that  $\Phi(1) < \infty$ , it holds that  $q = 0$ .

(ii) If  $a = 0$  and  $0 < b + c < \infty$ , then for any  $x \in \mathbf{R}_+$  and  $t > 0$

$$q(t, x) = x e^{-t(b+c)} + \frac{\delta}{b+c} (1 - e^{-t(b+c)}). \quad (2.10)$$

Suppose, in addition, that  $\Phi(1) < \infty$ . Then  $q = \delta/(b+c)$ .

*Proof.* As for  $q(t, x)$ , the calculation is based on (2.5) combined with a general fact that the infimum of the support of a probability measure  $\mu$  on  $\mathbf{R}_+$  is identified with the ‘low temperature limit’  $-\lim_{\lambda \rightarrow \infty} (d/d\lambda) \log \int f_\lambda d\mu$ . Thus

$$q(t, x) = \lim_{\lambda \rightarrow \infty} \left[ x \frac{\partial \psi}{\partial \lambda}(t, \lambda) + \int_0^t F'(\psi(s, \lambda)) \frac{\partial \psi}{\partial \lambda}(s, \lambda) ds \right]. \quad (2.11)$$

We introduce an auxiliary function  $R_0(\lambda) = -R(\lambda)/\lambda$ , which is positive and increasing for any  $\lambda > 0$ . It follows from (2.4) that for any  $t > 0$  and  $\lambda > 0$

$$t = - \int_{\psi(t, \lambda)}^\lambda \frac{du}{R(u)} = \int_{\psi(t, \lambda)}^\lambda \frac{du}{u R_0(u)}. \quad (2.12)$$

By differentiating this identity in  $\lambda$

$$\frac{\partial \psi}{\partial \lambda}(t, \lambda) = \frac{R(\psi(t, \lambda))}{R(\lambda)} = \frac{\psi(t, \lambda)}{\lambda} \cdot \frac{R_0(\psi(t, \lambda))}{R_0(\lambda)} \in (0, 1] \quad (2.13)$$

since  $\psi(t, \lambda) \leq \lambda$ . Also, noting that  $R_0(u) \in [R_0(\psi(t, \lambda)), R_0(\lambda)]$  for any  $u \in [\psi(t, \lambda), \lambda]$ , one can deduce from (2.12)

$$e^{-t R_0(\lambda)} \leq \frac{\psi(t, \lambda)}{\lambda} \leq e^{-t R_0(\psi(t, \lambda))}. \quad (2.14)$$

(i) To prove that  $q(t, x) = 0$ , we only need to show that  $\lim_{\lambda \rightarrow \infty} \frac{\partial \psi}{\partial \lambda}(t, \lambda) = 0$  for each  $t > 0$ . Indeed, noting that  $F'$  is decreasing and that  $\psi(s, \lambda)$  is decreasing in  $s$  and increasing in  $\lambda$  by (2.13), we have for any  $\lambda > 1$

$$0 \leq \int_0^t F'(\psi(s, \lambda)) \frac{\partial \psi}{\partial \lambda}(s, \lambda) ds \leq F'(\psi(t, 1)) \int_0^t \frac{\partial \psi}{\partial \lambda}(s, \lambda) ds.$$

By combining (2.13) with (2.14)

$$0 \leq \frac{\partial \psi}{\partial \lambda}(t, \lambda) \leq e^{-tR_0(\psi(t, \lambda))} \frac{R_0(\psi(t, \lambda))}{R_0(\lambda)} \leq \frac{1}{tR_0(\lambda)},$$

which tends to 0 as  $\lambda \rightarrow \infty$  since the assumption implies that  $R_0(\lambda) \rightarrow \infty$ . Consequently  $\frac{\partial \psi}{\partial \lambda}(t, \lambda)$  converges to 0 boundedly and by virtue of (2.11)  $q(t, x) = 0$ .

In the case where  $\Phi(1) < \infty$ ,  $q = \lim_{\lambda \rightarrow \infty} \Phi'(\lambda) = \lim_{\lambda \rightarrow \infty} F(\lambda)/(\lambda R_0(\lambda))$ . It is easy to see that  $\lim_{\lambda \rightarrow \infty} F(\lambda)/\lambda = \delta$ . So we conclude that  $q = 0$ .

(ii) It is obvious that the proof of (2.10) can be reduced to showing the following two asymptotics; as  $\lambda \rightarrow \infty$

$$\frac{\partial \psi}{\partial \lambda}(s, \lambda) \rightarrow e^{-(b+c)s} \quad \text{for each } s > 0$$

and

$$F'(\psi(s, \lambda)) \rightarrow \delta \quad \text{locally boundedly in } s \geq 0.$$

Observe that  $R_0(\lambda) \rightarrow b+c \in (0, \infty)$  by the assumption. Hence the first inequality in (2.14) implies that  $\psi(s, \lambda) \rightarrow \infty$ . By (2.14) again we have  $\psi(s, \lambda)/\lambda \rightarrow \exp(-(b+c)s)$  and thus (2.13) proves the first asymptotics. The second one is a consequence of the following estimate; for any  $s \in [0, t]$

$$|F'(\psi(s, \lambda)) - \delta| = \int_0^\infty y e^{-\psi(s, \lambda)y} n_2(dy) \leq \int_0^\infty y e^{-\psi(t, \lambda)y} n_2(dy).$$

Therefore (2.10) has been established.

The last part of the assertion (ii) can be shown by  $F(\lambda)/\lambda \rightarrow \delta$  and  $R_0(\lambda) \rightarrow b+c$  together. The proof of the proposition is complete.  $\blacksquare$

It is worth mentioning the diffusion case, namely the case where  $a, b > 0$  and  $n_1 \equiv 0 \equiv n_2$ . Because of (2.3) we have  $\delta > 0$ , and by (2.7) the corresponding CBI-process, known also as the CIR model, has a unique stationary distribution with Laplace exponent

$$\Phi(\lambda) = \frac{\delta}{a} \log \left( 1 + \frac{a}{b} \lambda \right) = \frac{\delta}{a} \int_0^\infty (1 - e^{-\lambda y}) \frac{e^{-by/a}}{y} dy.$$

It is a gamma distribution with parameter  $(\delta/a, b/a)$ , which has, by definition, the density proportional to  $x^{\delta/a-1} \exp(-bx/a)$ . This stationary distribution is reversible. In other words, the associated bilinear form (1.1) is symmetric.

It would be natural to ask if there is any other case which admits a nondegenerate reversible distribution. The following theorem, the main result of this section, gives a negative answer to this question.

**Theorem 2.3** *If the CBI-process with generator (2.1) has a nondegenerate reversible distribution, then the process coincides with a CIR model.*

Before proving this theorem we prepare a simple lemma. In what follows, the notation  $\langle f \rangle$  or  $\langle f(x) \rangle$  will stand for the integral  $\int_{\mathbf{R}_+} f(x) \nu(dx)$  with respect to the stationary distribution  $\nu$  of an ergodic CBI-process.

**Lemma 2.4** *Let  $\lambda, \mu \geq 0$  be arbitrary.*

(i) *For any  $x \in \mathbf{R}_+$*

$$-L f_\lambda(x) = (R(\lambda)x + F(\lambda)) f_\lambda(x). \quad (2.15)$$

(ii) *If the CBI-process has a stationary distribution, then*

$$\langle (-L) f_\lambda \cdot f_\mu \rangle = R(\lambda) (\Phi'(\lambda + \mu) - \Phi'(\lambda)) \langle f_{\lambda+\mu} \rangle. \quad (2.16)$$

*Proof.* (2.15) is verified by direct calculations. Using it, we have

$$\langle (-L) f_\lambda \cdot f_\mu \rangle = R(\lambda) \langle x e^{-(\lambda+\mu)x} \rangle + F(\lambda) \langle e^{-(\lambda+\mu)x} \rangle. \quad (2.17)$$

Also, (2.6) and (2.7) give  $\langle x e^{-(\lambda+\mu)x} \rangle = \Phi'(\lambda + \mu) \langle e^{-(\lambda+\mu)x} \rangle$  and  $F(\lambda) = -R(\lambda) \Phi'(\lambda)$ , respectively. (2.16) follows by plugging these equalities into (2.17). ■

*Proof of Theorem 2.3.* By the assumption, we have the symmetry of the Dirichlet form. In particular,  $\langle (-L) f_\lambda \cdot f_\mu \rangle = \langle (-L) f_\mu \cdot f_\lambda \rangle$  for all  $\lambda, \mu > 0$ . By virtue of Lemma 2.4 (ii), this becomes

$$R(\lambda) (\Phi'(\lambda + \mu) - \Phi'(\lambda)) = R(\mu) (\Phi'(\lambda + \mu) - \Phi'(\mu)).$$

Because of (2.7), the above equality is rewritten into

$$(R(\lambda) - R(\mu)) F(\lambda + \mu) = (F(\lambda) - F(\mu)) R(\lambda + \mu). \quad (2.18)$$

Differentiating in  $\lambda$  yields

$$R'(\lambda) F(\lambda + \mu) + (R(\lambda) - R(\mu)) F'(\lambda + \mu) = F'(\lambda) R(\lambda + \mu) + (F(\lambda) - F(\mu)) R'(\lambda + \mu).$$

By interchanging the roles of  $\lambda$  and  $\mu$

$$R'(\mu) F(\lambda + \mu) + (R(\mu) - R(\lambda)) F'(\lambda + \mu) = F'(\mu) R(\lambda + \mu) + (F(\mu) - F(\lambda)) R'(\lambda + \mu).$$

Summing up the above two equalities, we arrive at

$$(R'(\lambda) + R'(\mu)) F(\lambda + \mu) = (F'(\lambda) + F'(\mu)) R(\lambda + \mu). \quad (2.19)$$

Now let  $\lambda \neq \mu$ . Then (2.18) and (2.19) together with Cauchy's mean value theorem imply further that for some  $\xi$  between  $\lambda$  and  $\mu$

$$\frac{R'(\lambda) + R'(\mu)}{F'(\lambda) + F'(\mu)} = \frac{R(\lambda) - R(\mu)}{F(\lambda) - F(\mu)} = \frac{R'(\xi)}{F'(\xi)}. \quad (2.20)$$

Here it should be noted that by (2.3)

$$F'(u) = \delta + \int_0^\infty y e^{-uy} n_2(dy) > 0 \quad (2.21)$$



is convex and

$$R'(u) = -2au - b - \int_0^\infty y(1 - e^{-uy})n_1(dy) < 0$$

is convex. Therefore, (2.20) is possible only in the case where neither  $F'$  nor  $-R'$  is strictly convex. Consequently, both  $n_1$  and  $n_2$  must vanish. So (2.21) shows that  $\delta > 0$ , and the positivity of  $b$  is necessary for  $\Phi(\lambda)$  to be finite. Moreover,  $a$  must be positive also since otherwise  $\Phi(\lambda) = \delta\lambda/b$  implying that the stationary distribution is concentrated at  $\delta/b$ . The proof of Theorem 2.3 is complete.  $\blacksquare$

Here are concrete examples of CBI-processes with non-reversible stationary distributions.

*Example 2.1* (i) This example is taken from Example 4.2 of [14]. Given  $0 < \alpha < \beta < 1$ , set  $a = b = \delta = 0$ ,

$$n_1(dy) = \frac{\alpha(\alpha+1)}{\Gamma(1-\alpha)} \cdot \frac{dy}{y^{2+\alpha}} \quad \text{and} \quad n_2(dy) = \frac{\beta}{\Gamma(1-\beta)} \cdot \frac{dy}{y^{1+\beta}}.$$

Then  $R(\lambda) = -\lambda^{1+\alpha}$  and  $F(\lambda) = \lambda^\beta$ . Therefore,  $\Phi(\lambda) = \lambda^{\beta-\alpha}/(\beta-\alpha)$ , the Laplace exponent of a  $(\beta-\alpha)$ -stable distribution on  $\mathbf{R}_+$ . (2.16) gives

$$\langle (-L)f_\lambda \cdot f_\mu \rangle = \lambda^\beta \left\{ 1 - \left( \frac{\lambda}{\lambda + \mu} \right)^{1-(\beta-\alpha)} \right\} e^{-\Phi(\lambda+\mu)}.$$

(ii) Given  $0 < \alpha < 1$  and  $\kappa \geq 0$ , define  $a = 0$ ,  $b = \kappa^\alpha$ ,  $\delta = 1$ ,  $n_2 \equiv 0$  and

$$n_1(dy) = -\frac{\alpha}{\Gamma(1-\alpha)} \left( \frac{e^{-\kappa y}}{y^{1+\alpha}} \right)' dy = \frac{\alpha}{\Gamma(1-\alpha)} (\kappa y + 1 + \alpha) \frac{e^{-\kappa y}}{y^{2+\alpha}} dy. \quad (2.22)$$

With these choices  $R(\lambda) = -\lambda(\lambda + \kappa)^\alpha$  and  $F(\lambda) = \lambda$ , which together lead to

$$\Phi(\lambda) = \frac{1}{1-\alpha} [(\lambda + \kappa)^{1-\alpha} - \kappa^{1-\alpha}] = \frac{1}{\Gamma(\alpha)} \int_0^\infty (1 - e^{-\lambda y}) \frac{e^{-\kappa y}}{y^{1+(1-\alpha)}} dy. \quad (2.23)$$

See e.g. [5] for information of the corresponding distribution. Note that for  $\kappa = 0$  the stationary distribution is a  $(1-\alpha)$ -stable distribution on  $\mathbf{R}_+$ , and that as  $\alpha \uparrow 1$ ,  $\Phi(\lambda)$  tends to  $\log(1 + \lambda/\kappa)$ , the Laplace exponent of a gamma distribution, provided that  $\kappa > 0$ . By (2.16) we have

$$\langle (-L)f_\lambda \cdot f_\mu \rangle = \lambda \left\{ 1 - \left( \frac{\lambda + \kappa}{\lambda + \mu + \kappa} \right)^\alpha \right\} e^{-\Phi(\lambda+\mu)}.$$

Since the class of CBI-processes studied so far seems too wide for one to obtain further consequences which are useful for our purpose, we will be obliged to make an additional restriction in the subsequent sections. In this regard, it must be remarked that the condition that  $n_1 \equiv 0$  makes the correspondence between  $(a, b, n_2, \delta)$  and  $(q, \Lambda)$  in (2.8) too simple as will be seen from the general observation below.

**Lemma 2.5** *Let  $a, b \geq 0$  and suppose that a measure  $n_2$  on  $(0, \infty)$  is non-zero. For  $\lambda > 0$ , set*

$$\Phi(\lambda) = \int_0^\lambda \frac{\int_0^\infty (1 - e^{-uy}) n_2(dy)}{au^2 + bu} du = \int_0^\lambda \frac{\int_0^\infty e^{-uy} \widetilde{n}_2(dy)}{au + b} du.$$

*Then for each  $\lambda > 0$*

$$\Phi(\lambda) = \begin{cases} \infty & (b = 0) \\ \int_0^\infty (1 - e^{-\lambda y}) \frac{\widetilde{n}_2(dy)}{by} & (a = 0, b > 0) \\ \int_0^\infty (1 - e^{-\lambda y}) \left( \int_0^y e^{bz/a} \widetilde{n}_2(dz) \right) \frac{e^{-by/a}}{ay} dy & (a > 0, b > 0). \end{cases}$$

The proof requires only ‘Fubini calculus’ and so is left to the reader. In the light of this lemma, we shall proceed under the additional hypothesis that  $n_2 \equiv 0$ .

### 3 Estimating the sector constant

The main subject of this section is the estimation of the Dirichlet form. As announced in Introduction, we now show in a general setting that  $\text{Sect}(\mathcal{E}) > 1$  holds for any non-symmetric Dirichlet form  $\mathcal{E}$ . (An explicit lower bound for  $\text{Sect}(\mathcal{E})$  will be discussed at the end of this section.)

**Proposition 3.1** *Suppose that the bilinear form  $\mathcal{E}$  in (1.1) is associated with a conservative Markov process with generator  $L$  and a stationary distribution  $\nu$ . If  $\mathcal{E}$  is non-symmetric, then  $\text{Sect}(\mathcal{E}) > 1$ .*

*Proof.* We may assume that  $\text{Sect}(\mathcal{E}) < \infty$ . Equivalently, suppose that (1.2) holds for some  $C < \infty$ . By non-symmetry there exist  $f, g \in D(L)$  such that  $\mathcal{E}(f, g) > \mathcal{E}(g, f)$ . This implies  $\mathcal{E}(f, f) > 0$  since otherwise (1.2) leads to the contradiction that  $\mathcal{E}(f, g) = \mathcal{E}(g, f) (= 0)$ . It is straightforward to see that

$$\lim_{t \rightarrow 0} \frac{1}{t} \left( \frac{\mathcal{E}(f, f + tg)^2}{\mathcal{E}(f, f) \mathcal{E}(f + tg, f + tg)} - 1 \right) = \frac{\mathcal{E}(f, g) - \mathcal{E}(g, f)}{\mathcal{E}(f, f)} > 0, \quad (3.1)$$

and hence  $\mathcal{E}(f, f + tg)^2 > \mathcal{E}(f, f) \mathcal{E}(f + tg, f + tg)$  for  $t > 0$  small enough. This shows that  $\text{Sect}(\mathcal{E}) > 1$ . ■

We now turn to discussing the bilinear forms  $\mathcal{E}$  associated with ergodic CBI-processes. The symmetric part  $\tilde{\mathcal{E}}(f, g) := (\mathcal{E}(f, g) + \mathcal{E}(g, f))/2$  has an expression of the form

$$\begin{aligned} \tilde{\mathcal{E}}(f, g) &= a \langle x f'(x) g'(x) \rangle + \frac{1}{2} \langle x \int n_1(dy) (f(x+y) - f(x))(g(x+y) - g(x)) \rangle \\ &\quad + \frac{1}{2} \langle \int n_2(dy) (f(x+y) - f(x))(g(x+y) - g(x)) \rangle. \end{aligned} \quad (3.2)$$

Here and in what follows, the domain of integration may be suppressed as long as it is  $(0, \infty)$ . (3.2) can be verified by calculating  $\Gamma(f, g) := (L(fg) - Lf \cdot g - f \cdot Lg)/2$  since  $\tilde{\mathcal{E}}(f, g) = \langle \Gamma(f, g) \rangle$ . The main task in the remainder of this section is to give an upper bound of  $\text{Sect}(\mathcal{E})$  for a class of non-reversible CBI-processes.

In the rest of the paper, we make the restriction that  $n_2 \equiv 0$  and call such processes continuous state branching processes with continuous immigration, henceforth abbreviated as CBCI-processes. Thus,  $F(\lambda) = \delta\lambda$  with some  $\delta > 0$ . This condition seems crucial in the subsequent argument, in particular, in showing an integration by parts formula described in Proposition 3.2 below. The notation  $n$  is used instead of  $n_1$  and thus a measure  $n$  on  $(0, \infty)$  is assumed to satisfy  $\int \min\{y^2, y\}n(dy) < \infty$  according to (2.2). In the discussion below, we shall suppose the existence of a unique stationary distribution and introduce a one-parameter family of the convolution semigroup  $\{\nu_\delta := \nu^{*\delta}\}_{\delta>0}$  with  $\nu$  having the Laplace exponent

$$\Phi(\lambda) = \int_0^\lambda \frac{du}{au + b + \int (1 - e^{-uy})\tilde{n}(dy)} = q\lambda + \int (1 - e^{-\lambda y})\Lambda(dy), \quad (3.3)$$

where  $q \geq 0$  and  $\Lambda$  is a Lévy measure on  $(0, \infty)$ . Accordingly,  $\delta\Phi$  is the Laplace exponent of  $\nu_\delta$ , which is a unique stationary distribution of the process with generator

$$\begin{aligned} L_\delta f(x) &:= axf''(x) - bxf'(x) + x \int [f(x+y) - f(x) - yf'(x)]n(dy) \\ &\quad + \delta f'(x). \end{aligned}$$

We call it the CBCI-process with quadruplet  $(a, b, n, \delta)$ . This subclass of CBI-processes is one-dimensional version of the model discussed in [20] and [21]. We emphasize that an explicit formula for the Lévy density  $d\Lambda/dy$  was obtained in Lemma 2.5 of [21] under the additional hypothesis that  $a, b > 0$  and  $c = \int \tilde{n}(dy) < \infty$ . (In this case  $q = 0$  by Proposition 2.2 (i).) An analogue of that formula is available also in the case where  $a = 0$  and  $0 < b + c < \infty$ . Indeed, by differentiating (3.3)

$$\Phi'(\lambda) = \frac{1}{b + c - \int e^{-\lambda y}\tilde{n}(dy)} = \frac{1}{b + c} + \sum_{N=1}^{\infty} \frac{1}{b + c} \left( \frac{1}{b + c} \int e^{-\lambda y}\tilde{n}(dy) \right)^N,$$

which shows that  $q = 1/(b + c)$  and

$$\Lambda(dy) = \frac{1}{y} \sum_{N=1}^{\infty} \frac{1}{(b + c)^{N+1}} \tilde{n}^{*N}(dy). \quad (3.4)$$

The notation  $\langle \cdot \rangle_\delta$  will stand for the integral with respect to  $\nu_\delta$  and the associated bilinear form is denoted by  $\mathcal{E}^\delta$ , namely  $\mathcal{E}^\delta(f, g) = \langle (-L_\delta)f \cdot g \rangle_\delta$ . Let  $\mathcal{F}_0$  be the linear hull of  $\{f_\lambda : \lambda \geq 0\}$ . As remarked after Theorem 1.1' in [7],  $\mathcal{F}_0$  is a core of the generator of the CBI-process.

**Proposition 3.2** *Let  $\Phi$  be defined by the first equality in (3.3) and suppose that  $\Phi(1) < \infty$ . Then for each  $\delta > 0$*

$$\begin{aligned} \mathcal{E}^\delta(f, g) &= a \langle x f'(x) g'(x) \rangle_\delta \\ &\quad + \langle x \int \tilde{n}(dy) f'(x+y)(g(x+y) - g(x)) \rangle_\delta, \quad f, g \in \mathcal{F}_0. \end{aligned} \quad (3.5)$$

*Proof.* It suffices to show (3.5) for  $f = f_\lambda$ ,  $g = f_\mu$  with  $\lambda, \mu > 0$  being arbitrary. We begin with a version of (2.16) in Lemma 2.4:

$$\langle (-L_\delta) f_\lambda \cdot f_\mu \rangle_\delta = \delta \Phi'(\lambda + \mu) \langle f_{\lambda+\mu} \rangle_\delta \left( R(\lambda) + \frac{\lambda}{\Phi'(\lambda + \mu)} \right), \quad (3.6)$$

where  $R(\lambda) = -a\lambda^2 - b\lambda - \lambda \int \tilde{n}(dy)(1 - e^{-\lambda y})$ . Observing from (3.3) that

$$\frac{1}{\Phi'(\lambda + \mu)} = a(\lambda + \mu) + b + \int \tilde{n}(dy)(1 - e^{-(\lambda+\mu)y}),$$

we have

$$R(\lambda) + \frac{\lambda}{\Phi'(\lambda + \mu)} = a\lambda\mu + \lambda \int \tilde{n}(dy) (e^{-\lambda y} - e^{-(\lambda+\mu)y}). \quad (3.7)$$

Since  $\delta \Phi'(\lambda + \mu) \langle f_{\lambda+\mu} \rangle_\delta = \langle x f_{\lambda+\mu}(x) \rangle_\delta$ , we get by plugging (3.7) into (3.6)

$$\begin{aligned} \langle (-L_\delta) f_\lambda \cdot f_\mu \rangle_\delta &= \langle x f_{\lambda+\mu}(x) \rangle_\delta \left( a\lambda\mu + \lambda \int \tilde{n}(dy) (e^{-\lambda y} - e^{-(\lambda+\mu)y}) \right) \\ &= a \langle x f_\lambda'(x) f_\mu'(x) \rangle_\delta + \langle x \int \tilde{n}(dy) (-\lambda) e^{-\lambda(x+y)} (e^{-\mu(x+y)} - e^{-\mu x}) \rangle_\delta. \end{aligned} \quad (3.8)$$

This coincides with the right side of (3.5) with  $f_\lambda$  and  $f_\mu$  in place of  $f$  and  $g$ , respectively.  $\blacksquare$

The integration by parts formula (3.5) would be interesting in its own right and applicable in other contexts. We here use it for the purpose of the sector constant estimate. While it is obvious from (3.2) that the first term on the right side of (3.5) is dominated by  $\mathcal{E}^\delta(f, f)^{1/2} \mathcal{E}^\delta(g, g)^{1/2}$ , the main difficulty in handling the second term comes from the fact that we have few information on the distribution function (or the density function) of the stationary distribution  $\nu_\delta$ . We overcome this by a strategy similar to that taken in [21] for the proof of a Poincaré type inequality. That is, we show first that an estimate we want for the second term to satisfy can reduce to an analogous one for the Lévy measure  $\Lambda$ , and then give a sufficient condition for the reduced estimate to hold. So we shall be concerned with the bilinear forms  $B_\delta(\delta > 0)$  and  $B_0$  on  $\mathcal{F}_0 \times \mathcal{F}_0$  defined by

$$B_\delta(f, g) = \langle x \int \tilde{n}(dy) f'(x+y)(g(x+y) - g(x)) \rangle_\delta, \quad \delta > 0$$

and

$$B_0(f, g) = \int \Lambda(dx) x \int \tilde{n}(dy) f'(x+y)(g(x+y) - g(x)),$$

respectively. The first step will be done in the next theorem, the main technical result of this section.

**Theorem 3.3** *Let  $\Phi$  be defined by the first equality in (3.3). Suppose that  $\Phi(1) < \infty$  and that a Lévy measure  $\Lambda$  on  $(0, \infty)$  satisfies (3.3) with  $q = 0$ . Then, for any fixed  $0 < C < \infty$ , the following two conditions are equivalent to each other:*

(i) *For all  $\delta > 0$*

$$B_\delta(f, g)^2 \leq C \langle x f'(x)^2 \rangle_\delta \langle x \int n(dy) (g(x+y) - g(x))^2 \rangle_\delta, \quad f, g \in \mathcal{F}_0. \quad (3.9)$$

(ii) *For all  $f, g \in \mathcal{F}_0$*

$$B_0(f, g)^2 \leq C \int \Lambda(dx) x f'(x)^2 \cdot \int \Lambda(dx) x \int n(dy) (g(x+y) - g(x))^2. \quad (3.10)$$

If, in addition,  $a > 0$  and (3.9) holds for some  $\delta > 0$ , then  $\text{Sect}(\mathcal{E}^\delta) \leq 1 + \sqrt{2C/a}$ .

*Proof.* The implication (i)  $\implies$  (ii) follows immediately by observing that as  $\delta \downarrow 0$

$$\delta^{-1} \langle x h_i(x) \rangle_\delta = \int z \Lambda(dz) \langle h_i(x+z) \rangle_\delta \rightarrow \int z \Lambda(dz) h_i(z) \quad (3.11)$$

for each  $i \in \{1, 2, 3\}$ , where  $h_1(x) = \int \tilde{n}(dy) f'(x+y)(g(x+y) - g(x))$ ,  $h_2(x) = f'(x)^2$  and  $h_3(x) = \int n(dy) (g(x+y) - g(x))^2$  with  $f, g \in \mathcal{F}_0$  being arbitrary. In (3.11) we have applied the Palm formula for the underlying Poisson random measure to get the equality, and then used the fact that  $\nu_\delta$  tends weakly to the delta distribution at 0. (Alternatively, the equality can be verified directly for  $h_i = f_\lambda$  with  $\lambda > 0$  and extended easily. See Lemma 3.2 in [21].)

Next, assume that (3.10) holds for all  $f, g \in \mathcal{F}_0$ . We must show (3.9) for every  $\delta > 0$ . Let  $f$  and  $g$  be given as finite sums of the form  $f = \sum_i c_i f_{\lambda_i}$  and  $g = \sum_j d_j f_{\mu_j}$ , respectively, where  $c_i, d_j \in \mathbf{R}$ ,  $\lambda_i, \mu_j \geq 0$ . In view of (3.8)

$$\begin{aligned} B_\delta(f_\lambda, f_\mu) &= \langle x f_{\lambda+\mu}(x) \rangle_\delta \lambda \int \tilde{n}(dy) (e^{-\lambda y} - e^{-(\lambda+\mu)y}) \\ &= \delta e^{-\delta\Phi(\lambda+\mu)} \Phi'(\lambda+\mu) \int \tilde{n}(dy) \lambda (e^{-\lambda y} - e^{-(\lambda+\mu)y}) \end{aligned}$$

for any  $\lambda, \mu \geq 0$ . Therefore, by bilinearity  $B_\delta(f, g)$  equals

$$\begin{aligned} &\delta \sum_{i,j} c_i d_j e^{-\delta\Phi(\lambda_i+\mu_j)} \Phi'(\lambda_i+\mu_j) \int \tilde{n}(dy) \lambda_i (e^{-\lambda_i y} - e^{-(\lambda_i+\mu_j)y}) \\ &= \delta \sum_{i,j} c'_i d'_j e^{\delta\Phi(\lambda_i, \mu_j)} \int \Lambda(dx) x e^{-(\lambda_i+\mu_j)x} \int \tilde{n}(dy) \lambda_i (e^{-\lambda_i y} - e^{-(\lambda_i+\mu_j)y}), \end{aligned} \quad (3.12)$$

where  $c'_i = c_i \exp(-\delta\Phi(\lambda_i))$ ,  $d'_j = d_j \exp(-\delta\Phi(\mu_j))$  and

$$\Phi(\lambda, \mu) = \Phi(\lambda) + \Phi(\mu) - \Phi(\lambda + \mu) = \int \Lambda(dz) (1 - e^{-\lambda z})(1 - e^{-\mu z}), \quad \lambda, \mu \geq 0.$$

Substituting the expansion

$$e^{\delta\Phi(\lambda_i, \mu_j)} = \sum_{N=0}^{\infty} \frac{\delta^N}{N!} \int \Lambda(dx_1) \cdots \int \Lambda(dx_N) \prod_{k=1}^N (1 - e^{-\lambda_i x_k}) \prod_{l=1}^N (1 - e^{-\mu_j x_l})$$

into (3.12) leads to

$$B_\delta(f, g) = \delta \sum_{N=0}^{\infty} \frac{\delta^N}{N!} \int \Lambda(dx_1) \cdots \int \Lambda(dx_N) \int \Lambda(dx) x \int \tilde{n}(dy) f_{x_1, \dots, x_N}'(x+y) (g_{x_1, \dots, x_N}(x+y) - g_{x_1, \dots, x_N}(x)),$$

where

$$f_{x_1, \dots, x_N}(x) = \sum_i c'_i \prod_{k=1}^N (1 - e^{-\lambda_i x_k}) f_{\lambda_i}(x)$$

and

$$g_{x_1, \dots, x_N}(x) = \sum_j d'_j \prod_{l=1}^N (1 - e^{-\mu_j x_l}) f_{\mu_j}(x)$$

are considered to be elements of  $\mathcal{F}_0$  for arbitrarily given  $x_1, \dots, x_N > 0$ . We can apply now (3.10) to these functions and then use Schwarz's inequality to obtain

$$B_\delta(f, g) = \delta \sum_{N=0}^{\infty} \frac{\delta^N}{N!} \int \Lambda(dx_1) \cdots \int \Lambda(dx_N) B_0(f_{x_1, \dots, x_N}, g_{x_1, \dots, x_N}) \quad (3.13)$$

$$\leq \sqrt{C} \sqrt{Q_\delta^{(1)}(f)} \sqrt{Q_\delta^{(2)}(g)}, \quad (3.14)$$

where

$$Q_\delta^{(1)}(f) = \delta \sum_{N=0}^{\infty} \frac{\delta^N}{N!} \int \Lambda(dx_1) \cdots \int \Lambda(dx_N) \int \Lambda(dx) x f_{x_1, \dots, x_N}'(x)^2$$

and  $Q_\delta^{(2)}(g)$  is defined to be

$$\delta \sum_{N=0}^{\infty} \frac{\delta^N}{N!} \int \Lambda(dx_1) \cdots \int \Lambda(dx_N) \int \Lambda(dx) x \int n(dy) (g_{x_1, \dots, x_N}(x+y) - g_{x_1, \dots, x_N}(x))^2.$$

But analogous calculations to those for the proof of (3.13) show that

$$Q_\delta^{(1)}(f) = \langle x f'(x)^2 \rangle_\delta \quad \text{and} \quad Q_\delta^{(2)}(g) = \langle x \int n(dy) (g(x+y) - g(x))^2 \rangle_\delta.$$

Thus, (3.14) proves (3.9).

Lastly, in view of (3.5) and (3.2) with  $n_1 = n$  and  $n_2 \equiv 0$ , the validity of (3.9) implies that

$$\mathcal{E}^\delta(f, g) \leq (1 + \sqrt{2C/a}) \mathcal{E}^\delta(f, f)^{1/2} \mathcal{E}^\delta(g, g)^{1/2}, \quad f, g \in \mathcal{F}_0$$

if  $a > 0$ . This inequality is shown to extend to all functions in  $D(L_\delta)$ , and hence  $\text{Sect}(\mathcal{E}^\delta) \leq 1 + \sqrt{2C/a}$  as desired. The proof of Theorem 3.3 is complete.  $\blacksquare$

The next step is to seek conditions for (3.10) to hold. Define

$$\left\| \frac{d\tilde{n}}{dn} \right\|_\infty = \inf \{ r > 0 : \tilde{n}(dy) \leq r n(dy) \text{ in distribution sense} \}$$

with convention that  $\inf \emptyset = \infty$ . Clearly, this value is 0 for  $n \equiv 0$ .

**Corollary 3.4** *In addition to the assumptions in Theorem 3.3, assume that  $a > 0$  and that there exists a density  $d\Lambda/dy =: \varphi(y)/y$  such that*

$$(\varphi * \tilde{n})(z) := \int_0^z \varphi(z-y) \tilde{n}(dy) \leq C_1 \varphi(z), \quad z > 0 \quad (3.15)$$

for some  $0 \leq C_1 < \infty$ . Then for any  $\delta > 0$

$$\text{Sect}(\mathcal{E}^\delta) \leq 1 + \sqrt{\frac{2C_1}{a} \left\| \frac{d\tilde{n}}{dn} \right\|_\infty}. \quad (3.16)$$

*Proof.* We may assume that  $\|d\tilde{n}/dn\|_\infty < \infty$ . By virtue of Theorem 3.3, it is enough to show (3.10) with  $C = C_1 r$  for any  $r > 0$  such that  $\tilde{n}(dy) \leq rn(dy)$ . Applying Schwarz's inequality, we can dominate  $B_0(f, g)^2$  by

$$\begin{aligned} & \int \Lambda(dx) x \int \tilde{n}(dy) f'(x+y)^2 \cdot \int \Lambda(dx) x \int \tilde{n}(dy) (g(x+y) - g(x))^2 \\ & \leq \int dx \varphi(x) \int \tilde{n}(dy) f'(x+y)^2 \cdot r \int \Lambda(dx) x \int n(dy) (g(x+y) - g(x))^2. \end{aligned}$$

Since by (3.15)

$$\begin{aligned} \int dx \varphi(x) \int \tilde{n}(dy) f'(x+y)^2 &= \int dz f'(z)^2 (\varphi * \tilde{n})(z) \\ &\leq C_1 \int dz f'(z)^2 \varphi(z) = C_1 \int \Lambda(dz) z f'(z)^2, \end{aligned}$$

the desired inequality is derived. ■

A key ingredient to verify (3.15) is the following fact taken from Eq. (6) in the proof of Lemma 2.6 in [21]. (The function  $K$  there is identical with  $\varphi$  in the present paper.)

**Lemma 3.5** *In addition to the assumptions in Theorem 3.3, assume that  $a > 0$  and  $0 < \tilde{b} := b + c < \infty$ . Then the Lévy measure  $\Lambda$  in (3.3) has a strictly positive density  $d\Lambda/dy =: \varphi(y)/y$  with  $\varphi$  being differentiable. Moreover,  $\varphi(0) := \lim_{y \downarrow 0} \varphi(y) = 1/a$  and*

$$(\varphi * \tilde{n})(y) = a\varphi'(y) + \tilde{b}\varphi(y), \quad y > 0. \quad (3.17)$$

To grasp the validity of (3.17), it is worth noting that taking the Laplace transform of both side of (3.17) leads, at least at formal level, to the equation equivalent to the one derived by differentiating (3.3) with  $q = 0$  provided that  $\varphi(0) = 1/a$ . Note also that  $\varphi'(0) := \lim_{y \downarrow 0} \varphi'(y) = -\tilde{b}/a^2$  is deduced from (3.17).

**Proposition 3.6** *Under the same assumptions and with the same notation as in Lemma 3.5, define  $V(y) = -\log \varphi(y)$  (, so that  $\lim_{y \downarrow 0} V'(y) =: V'(0)$  exists). If*

$$\sup\{V'(0) - V'(y) : y > 0\} \leq C_2$$

for some  $0 \leq C_2 < \infty$ , then for any  $\delta > 0$

$$\text{Sect}(\mathcal{E}^\delta) \leq 1 + \sqrt{2C_2 \left\| \frac{d\tilde{n}}{dn} \right\|_\infty}. \quad (3.18)$$

*Proof.* By letting  $y \downarrow 0$  in (3.17)  $a\varphi'(0) + \tilde{b}\varphi(0) = 0$  or  $\tilde{b} = -a\varphi'(0)/\varphi(0) = aV'(0)$ . So again by (3.17)

$$(\varphi * \tilde{n})(y) = a\varphi'(y) + a\varphi(y)V'(0) = a\varphi(y)(-V'(y) + V'(0)) \leq aC_2\varphi(y),$$

and thus (3.15) with  $C_1 = aC_2$  holds true. Therefore (3.18) follows from (3.16).  $\blacksquare$

In the reversible case  $n \equiv 0$ , the function  $V$  in Proposition 3.6 is affine, so that we can take  $C_2 = 0$ . Under some integrability condition on  $n$ , quantitative information of  $C_2$  can be obtained from Lemma 2.6 in [21] combined with Eq. (6) there. It will turn out that (3.18) is one of basic tools in the next section, where more specific cases are discussed.

A naive guess based on the integration by parts formula (3.5) would be that  $\text{Sect}(\mathcal{E}) = \infty$  whenever  $a = 0$ . We have not proved this, nor given any sufficient condition for the CBCI-process not to satisfy the strong sector condition. We just present a simple (but very special) example of such a CBCI-process.

*Example 3.1* Let  $b, c > 0$  and  $n(dy) = c\epsilon_1(dy)$ , where  $\epsilon_1$  is the delta distribution at 1. Then, for each  $\delta > 0$ , the CBCI-process with quadruplet  $(0, b, n, \delta)$  is ergodic and does not satisfy the strong sector condition. Indeed, letting  $f(x) = \sin 2\pi x$  and  $g(x) = \cos 2\pi x$ , one can observe from (3.2) and (3.5) that  $\mathcal{E}^\delta(f, f) = 0 = \mathcal{E}^\delta(g, g)$  and that

$$\begin{aligned} \mathcal{E}^\delta(f, g) &= 2\pi c \langle x \int_0^1 \cos 2\pi(x+y)(\cos 2\pi(x+y) - \cos 2\pi x) dy \rangle_\delta \\ &= \pi c \langle x \rangle_\delta = \pi c \delta \Phi'(0) = \pi c \delta / b, \end{aligned}$$

respectively.

For later use, we close this section by giving a lower bound of  $\text{Sect}(\mathcal{E})$  in a general setting as a refinement of the calculation (3.1).

**Proposition 3.7** *Let  $\mathcal{E}$  be as in Proposition 3.1 and  $f, g \in D(L)$  be such that  $\mathcal{E}(f, f)\mathcal{E}(g, g) > 0$  and  $\tilde{\mathcal{E}}(f, g) := (\mathcal{E}(f, g) - \mathcal{E}(g, f))/2 > 0$ . Then*

$$\text{Sect}(\mathcal{E})^2 - 1 \geq \begin{cases} \frac{\tilde{\mathcal{E}}(f, g)^2}{\Delta(f, g)} & (\mathcal{E}(f, f)\mathcal{E}(g, g) \neq \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g), \Delta(f, g) > 0) \\ \infty & (\mathcal{E}(f, f)\mathcal{E}(g, g) \neq \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g), \Delta(f, g) = 0) \\ \frac{\tilde{\mathcal{E}}(f, g)}{\tilde{\mathcal{E}}(f, g)} & (\mathcal{E}(f, f)\mathcal{E}(g, g) = \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g)), \end{cases} \quad (3.19)$$

where  $\Delta(f, g) = \mathcal{E}(f, f)\mathcal{E}(g, g) - \tilde{\mathcal{E}}(f, g)^2 \geq 0$ .

*Proof.* The proof will be based on

$$\text{Sect}(\mathcal{E})^2 \geq \sup_{t \in \mathbf{R}} \frac{\mathcal{E}(f, f + tg)^2}{\mathcal{E}(f, f)\mathcal{E}(f + tg, f + tg)}. \quad (3.20)$$



Put  $U(t) = \mathcal{E}(f, f + tg)^2 / \mathcal{E}(f + tg, f + tg)$ . By direct calculations it can be shown that  $\frac{d}{dt} \log U(t)$  vanishes only for

$$t = t_0 := \frac{\mathcal{E}(f, f)(\mathcal{E}(f, g) - \tilde{\mathcal{E}}(f, g))}{\mathcal{E}(f, f)\mathcal{E}(g, g) - \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g)}$$

if  $\mathcal{E}(f, f)\mathcal{E}(g, g) - \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g) \neq 0$ , whereas  $U'(t)$  never vanishes unless  $U(t) = 0$  if  $\mathcal{E}(f, f)\mathcal{E}(g, g) - \mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g) = 0$ . In the former case, the supremum in (3.20) is achieved at  $t = t_0$  and (3.19) is obtained for the first two cases by calculating  $\lim_{t \rightarrow t_0} U(t)$ . In the latter case, noting that  $\mathcal{E}(f, g) = \tilde{\mathcal{E}}(f, g) + \check{\mathcal{E}}(f, g)$ , we have

$$\begin{aligned} \sup_{t \in \mathbf{R}} \frac{U(t)}{\mathcal{E}(f, f)} &= \lim_{|t| \rightarrow \infty} \frac{U(t)}{\mathcal{E}(f, f)} = \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(f, f)\mathcal{E}(g, g)} \\ &= \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(f, g)\tilde{\mathcal{E}}(f, g)} = 1 + \frac{\check{\mathcal{E}}(f, g)}{\tilde{\mathcal{E}}(f, g)}, \end{aligned}$$

which proves (3.19) for the third case. Lastly, the inequality  $\Delta(f, g) \geq 0$  holds in general since  $\tilde{\mathcal{E}}$  is symmetric and nonnegative definite.  $\blacksquare$

## 4 Generalized gamma convolutions as stationary distributions

In this section, we apply the results in the previous section to the sector constant estimate for a class of CBCI-processes whose stationary distributions are GGC's. (The general reference for GGC's is [2]. The interested reader is referred also to [22] or [6].) The situation, however, is a converse of that in Section 3 in the following sense. We shall be given a priori  $q \geq 0$  and the Lévy measure  $\Lambda$  of some GGC, and then intend to choose  $a, b$  and  $n$  so that (3.3) holds. To be more specific, recall that a GGC is an infinitely divisible distribution on  $\mathbf{R}_+$  with Lévy measure of the form

$$\Lambda_m(dy) := \left( \int e^{-uy} m(du) \right) \frac{dy}{y}, \quad (4.1)$$

where  $m$  (referred to as the Thorin measure) is a measure on  $(0, \infty)$  with

$$\int_{(0, 1/2]} |\log u| m(du) + \int_{(1/2, \infty)} u^{-1} m(du) < \infty.$$

This condition is necessary and sufficient for the Laplace exponent

$$\Phi_{q,m}(\lambda) := q\lambda + \int (1 - e^{-\lambda y}) \Lambda_m(dy) = q\lambda + \int \log \left( 1 + \frac{\lambda}{u} \right) m(du) \quad (4.2)$$

to be finite for all  $\lambda > 0$ . (Notice that found in the literature is the condition that  $\int_{(0, 1]} |\log u| m(du) + \int_{(1, \infty)} u^{-1} m(du) < \infty$ , which allows  $m(du) = \mathbf{1}_{(0, 1)}(u) du / |\log u|$

inappropriately. Here and in what follows, the notation  $\mathbf{1}_E$  denotes the indicator function of a set  $E$ .) We call the distribution having the Laplace exponent (4.2) the GGC with pair  $(q, m)$ . It must be remembered that every gamma distribution has a degenerate Thorin measure.

In order to study the above mentioned problem, we do a heuristic calculation; differentiate (3.3) with Lévy measure (4.1) to get

$$q + \int \frac{1}{\lambda + u} m(du) = \frac{1}{a\lambda + b + \int (1 - e^{-\lambda u}) \tilde{n}(du)}, \quad \lambda > 0. \quad (4.3)$$

This equation motivates us to exploit the theory of Bernstein functions [18]. Defining  $\mathcal{M}$  to be the totality of measures  $m$  on  $(0, \infty)$  such that  $\int (1 + u)^{-1} m(du) < \infty$ , we recall that every complete Bernstein function  $g$  is represented uniquely in the form

$$g(\lambda) = q\lambda + r + \int \frac{\lambda}{\lambda + u} m(du), \quad \lambda > 0 \quad (4.4)$$

where  $q, r \geq 0$  and  $m \in \mathcal{M}$  (cf. Remark 6.4 in [18]), and Proposition 7.1 in [18] asserts that a function  $g : (0, \infty) \rightarrow \mathbf{R}$  is a non-zero complete Bernstein function if and only if  $g^*(\lambda) := \lambda/g(\lambda)$  is a complete Bernstein function. With the help of these two facts one can show the next lemma, a key to our construction of the desired CBCI-process. In what follows we adopt the convention that  $1/\infty = 0$  and set  $\overline{m}_\alpha = \int u^\alpha m(du)$  for  $\alpha \in \mathbf{R}$ .

**Lemma 4.1** *Let  $q, r \geq 0$  and suppose that  $m \in \mathcal{M}$  is non-zero. Then there exist uniquely  $a, b \geq 0$  and  $M \in \mathcal{M}$  such that*

$$q + \frac{r}{\lambda} + \int \frac{1}{\lambda + u} m(du) = \frac{1}{a\lambda + b + \int \frac{\lambda}{\lambda + u} M(du)}, \quad \lambda > 0. \quad (4.5)$$

Moreover,

$$a = \begin{cases} 0 & (q > 0) \\ 1/(r + \overline{m}_0) & (q = 0), \end{cases} \quad b = \begin{cases} 0 & (r > 0) \\ 1/(q + \overline{m}_{-1}) & (r = 0) \end{cases} \quad (4.6)$$

and

$$\overline{M}_0 = \begin{cases} 1/q - b & (q > 0) \\ \overline{m}_1/(r + \overline{m}_0)^2 - b & (q = 0, \overline{m}_0 < \infty) \\ \infty & (q = 0, \overline{m}_0 = \infty). \end{cases} \quad (4.7)$$

*Proof.* The first half is immediate from the general facts on Bernstein functions previously mentioned. Indeed, defining the function  $g$  on  $(0, \infty)$  by (4.4), we can find uniquely  $a, b \geq 0$  and  $M \in \mathcal{M}$  such that

$$\frac{\lambda}{g(\lambda)} = g^*(\lambda) = a\lambda + b + \int \frac{\lambda}{\lambda + u} M(du), \quad \lambda > 0.$$

This is nothing but (4.5). Most calculations needed to show (4.6) and (4.7) are simple. While letting  $\lambda \rightarrow \infty$  in (4.5) yields  $a = 0$  whenever  $q > 0$ , letting  $\lambda \downarrow 0$  in (4.5) gives the value of  $b$  in (4.6). In the case  $q = 0$ , letting  $\lambda \rightarrow \infty$  in (4.5) multiplied by  $\lambda$  shows that  $a = 1/(r + \overline{m}_0)$ . With (4.6) in mind (4.7) can be proved in a similar manner because  $\overline{M}_0 = \lim_{\lambda \rightarrow \infty} \int \frac{\lambda}{\lambda+u} M(du)$ . For instance, in the case where  $q = 0$  and  $\overline{m}_0 < \infty$ , (4.5) and (4.6) together yield

$$\begin{aligned} \int \frac{\lambda}{\lambda+u} M(du) + b &= \frac{1}{\frac{r}{\lambda} + \int \frac{1}{\lambda+u} m(du)} - \frac{\lambda}{r + \overline{m}_0} \\ &= \frac{\int \frac{\lambda u}{\lambda+u} m(du)}{\left(r + \int \frac{\lambda}{\lambda+u} m(du)\right)(r + \overline{m}_0)}. \end{aligned}$$

Letting  $\lambda \rightarrow \infty$  proves (4.7) in this case. The proof for the other cases are left to the reader.  $\blacksquare$

Denote by  $\mathcal{S}(m)$  the support of a measure  $m$ . (4.7) together with Schwarz's inequality implies that  $M \equiv 0$  if and only if  $q = 0 = r$  and  $m$  is degenerate. In such a case, it is understood that  $\mathcal{S}(M) = \emptyset$ . The next lemma gives bounds of  $\inf \mathcal{S}(M)$  and  $\sup \mathcal{S}(M)$  in terms of  $q, r$  and  $m$ .

**Lemma 4.2** *Let  $q, r \geq 0$  and suppose that  $m \in \mathcal{M}$  is non-zero. Let  $a, b \geq 0$  and  $M \in \mathcal{M}$  be as in Lemma 4.1. Define  $s_- = s_-(q, r, m)$  and  $s_+ = s_+(q, r, m)$  by*

$$s_- = \sup \left\{ s < \inf \mathcal{S}(m) : s \left( q + \int \frac{m(du)}{u-s} \right) < r \right\}$$

and

$$s_+ = \inf \left\{ s > \sup \mathcal{S}(m) : s \left( q - \int \frac{m(du)}{s-u} \right) > r \right\},$$

respectively. Then the following assertions hold.

(i) *If  $a = 0 = b$ , then*

$$\frac{r}{r + \overline{m}_0 + q \inf \mathcal{S}(m)} \inf \mathcal{S}(m) \leq s_- \leq \inf \mathcal{S}(M)$$

and

$$\sup \mathcal{S}(M) \leq s_+ \begin{cases} \leq \sup \mathcal{S}(m) + (r + \overline{m}_0)q^{-1} & (q > 0) \\ = \infty & (q = 0). \end{cases}$$

(ii) *If  $a > 0$  and  $b = 0$ , then*

$$\frac{r}{r + \overline{m}_0} \inf \mathcal{S}(m) \leq s_- \leq \inf \mathcal{S}(M) \leq \sup \mathcal{S}(M) \leq \sup \mathcal{S}(m).$$

(iii) If  $a = 0$  and  $b > 0$ , then

$$\inf \mathcal{S}(m) \leq \inf \mathcal{S}(M) \leq \sup \mathcal{S}(M) \leq s_+ \begin{cases} \leq \sup \mathcal{S}(m) + \overline{m}_0 q^{-1} & (q > 0) \\ = \infty & (q = 0). \end{cases}$$

(iv) If  $a, b > 0$ , then  $\mathcal{S}(M)$  is contained in the closed interval from  $\inf \mathcal{S}(m)$  to  $\sup \mathcal{S}(m)$ .

Before going to the proof, it is worth noting that the analytic extension of (4.5) can be regarded as a relation between Stieltjes transforms of  $m$  and  $M$ . Introducing the notation  $G_m(z) = \int (z - u)^{-1} m(du)$  for  $m \in \mathcal{M}$ , we deduce from (4.5)

$$G_m(z) - q + \frac{r}{z} = \frac{1}{az - b - zG_M(z)}, \quad z \in \mathbf{C} \setminus [\inf \mathcal{S}(m), \sup \mathcal{S}(m)]. \quad (4.8)$$

*Proof of Lemma 4.2.* We shall employ the following fact. (See e.g. Theorem A.6 in [10].) For any  $s_1 < s_2$ , a measure  $M$  is supported on  $[s_1, s_2]$  if and only if  $G_M$  is holomorphic on  $\mathbf{C} \setminus [s_1, s_2]$ , negative on the interval  $(-\infty, s_1)$  and positive on the interval  $(s_2, \infty)$ .

To show (i) we assume, in addition to  $a = b = 0$ , that  $0 < \inf \mathcal{S}(m) \leq \sup \mathcal{S}(m) < \infty$ . By (4.8) we have

$$G_M(z) = \frac{1}{z \left( q - \int \frac{m(du)}{z - u} \right) - r} = \frac{-1}{r - z \left( q + \int \frac{m(du)}{u - z} \right)},$$

from which it is easily verified that  $G_M$  is holomorphic on  $\mathbf{C} \setminus [s_-, s_+]$ , positive for all  $z > s_+$  and negative for all  $z < s_-$ . So,  $s_- \leq \inf \mathcal{S}(M) \leq \sup \mathcal{S}(M) \leq s_+$  for the abovementioned reason. To prove the estimates for  $s_-$  and  $s_+$ , we may assume further that  $\overline{m}_0 < \infty$ . We then have  $q > 0$  because of (4.6). It is clear that a lower estimate for  $s_-$  is given by the smallest solution, say  $s'_- = s'_-(q, r, m)$ , to the quadratic equation in  $s$

$$s \left( q + \frac{\overline{m}_0}{\inf \mathcal{S}(m) - s} \right) = r,$$

namely

$$\begin{aligned} s_- &\geq s'_- \\ &= \frac{(r + \overline{m}_0 + q \inf \mathcal{S}(m)) - \sqrt{(r + \overline{m}_0 + q \inf \mathcal{S}(m))^2 - 4qr \inf \mathcal{S}(m)}}{2q} \\ &\geq \frac{r}{r + \overline{m}_0 + q \inf \mathcal{S}(m)} \inf \mathcal{S}(m). \end{aligned} \quad (4.9)$$

An upper estimate for  $s_+$  is given by the largest solution, say  $s'_+ = s'_+(q, r, m)$ , to

$$s \left( q + \frac{\overline{m}_0}{\sup \mathcal{S}(m) - s} \right) = r.$$

Thus  $s_+$  is dominated by

$$\begin{aligned} s'_+ &= \frac{(r + \overline{m}_0 + q \sup \mathcal{S}(m)) + \sqrt{(r + \overline{m}_0 + q \sup \mathcal{S}(m))^2 - 4qr \sup \mathcal{S}(m)}}{2q} \\ &\leq \sup \mathcal{S}(m) + \frac{r + \overline{m}_0}{q}. \end{aligned}$$

Also, it is obvious that  $s_+(= \inf \emptyset) = \infty$  for  $q = 0$ .

Next we prove (ii), assuming that  $a > 0$  and  $b = 0$ . By the former  $q = 0$  and  $\overline{m}_0 < \infty$ . Therefore, again by (4.8)

$$G_M(z) = \frac{\int \frac{u}{z-u} m(du)}{\left(r - z \int \frac{m(du)}{u-z}\right)(r + \overline{m}_0)}.$$

This allows proceeding along the same lines as the proof of (i). The proof of (iii) follows very closely the proof of (ii). So, these proofs are omitted. It remains to prove (iv). To this end, assume that  $a, b > 0$  and observe from (4.6) and (4.5) that

$$G_M(z) = \frac{\overline{m}_{-1} \int \frac{1}{z-u} m(du) - \overline{m}_0 \int \frac{1}{u(z-u)} m(du)}{\overline{m}_0 \overline{m}_{-1} \int \frac{1}{z-u} m(du)},$$

in which both  $\overline{m}_0$  and  $\overline{m}_{-1}$  are positive and finite. This shows the analyticity of  $G_M$  on  $\mathbf{C} \setminus [\inf \mathcal{S}(m), \sup \mathcal{S}(m)]$ . We may assume additionally that  $m$  is nondegenerate, for otherwise the assertion is trivial since  $M \equiv 0$ . Then the positivity of the numerator on the right side for any  $z \in \mathbf{R} \setminus [\inf \mathcal{S}(m), \sup \mathcal{S}(m)]$  follows from

$$\begin{aligned} &\overline{m}_{-1} \int \frac{1}{z-u} m(du) - \overline{m}_0 \int \frac{1}{u(z-u)} m(du) \\ &= \int \int \left( \frac{1}{v(z-u)} - \frac{1}{v(z-v)} \right) m(du) m(dv) \\ &= \int \frac{u}{z-u} m(du) \int \frac{1}{v(z-v)} m(dv) - \left( \int \frac{1}{z-u} m(du) \right)^2 > 0. \end{aligned}$$

These properties together prove (iv). ■

We now present the main result of this section, which concerns the construction of the CBCI-process having a given GGC as a stationary distribution. In other words, the problem (II) addressed in Introduction is solved. Simultaneously, the sector constant estimate will be obtained as a solution to the problem (III). Note that every Thorin measure belongs to  $\mathcal{M}$ .

**Theorem 4.3** *Let  $q \geq 0$  and suppose that  $m$  is a non-zero Thorin measure. Let  $a, b \geq 0$  and  $M \in \mathcal{M}$  be as in Lemma 4.1 with  $r = 0$ . Then the GGC with pair  $(q, m)$*

is a unique stationary distribution of the CBCI-process with quadruplet  $(a, b, n, 1)$ , where  $n$  is a measure on  $(0, \infty)$  defined to be

$$n(dy) = dy \int u^2 e^{-uy} M(du). \quad (4.10)$$

If, in addition,  $q = 0$ ,  $\overline{m}_1 < \infty$  and  $\inf \mathcal{S}(m) > 0$ , then  $\overline{m}_0 < \infty$ ,  $\inf \mathcal{S}(M) > 0$  and the CBCI-process with quadruplet  $(a, b, n, \delta)$  satisfies

$$\text{Sect}(\mathcal{E}^\delta) - 1 \leq \sqrt{\left(\frac{\overline{m}_1}{\overline{m}_0} - \inf \mathcal{S}(m)\right) \frac{2}{\inf \mathcal{S}(M)}} \leq \sqrt{2 \left(\frac{\overline{m}_1}{\overline{m}_0 \cdot \inf \mathcal{S}(m)} - 1\right)} \quad (4.11)$$

for any  $\delta > 0$ .

*Proof* Firstly, we claim  $\int \min\{y, y^2\} n(dy) < \infty$ . To see this, note that by (4.10) and Fubini's theorem

$$\begin{aligned} \int \min\{y, y^2\} n(dy) &= \int M(du) u^2 \int \min\{y, y^2\} e^{-uy} dy \\ &= \int M(du) \int y \min\{1, y/u\} e^{-y} dy \\ &= \int M(du) \left( \int_0^u (y^2/u) e^{-y} dy + \int_u^\infty y e^{-y} dy \right). \end{aligned} \quad (4.12)$$

It is elementary to verify that the integrand in the last expression is bounded above (and below) by a positive constant times  $1/(1+u)$ . Therefore it follows from  $M \in \mathcal{M}$  that  $\int \min\{y, y^2\} n(dy)$  is finite. Applying Fubini's theorem, we see that  $\int (1 - e^{-\lambda y}) \tilde{n}(dy) = \lambda \int (\lambda + u)^{-1} M(du)$  for  $\lambda > 0$ . Plugging this into (4.5) with  $r = 0$  yields (4.3). By integrating it

$$q\lambda + \int \log \left(1 + \frac{\lambda}{u}\right) m(du) = \int_0^\lambda \frac{du}{au + b + \int (1 - e^{-uy}) \tilde{n}(dy)}, \quad \lambda \geq 0.$$

In view of (3.3) and (4.2), this proves the first half.

For the proof of the last half, we assume additionally that  $q = 0$ ,  $\overline{m}_1 < \infty$  and  $\inf \mathcal{S}(m) > 0$ . The last two conditions together imply that  $\overline{m}_0, \overline{m}_{-1} < \infty$  since  $m$  is a Thorin measure. Hence  $a, b > 0$  by (4.6). Also, by (4.10) and (4.7)  $\int y n(dy) = \overline{M}_0 = \overline{m}_1/\overline{m}_0^2 - 1/\overline{m}_{-1}$ , which is finite. We now apply Proposition 3.6 to the Lévy measure (4.1) or equivalently to the function  $\varphi(y) := \int e^{-uy} m(du)$ . It is easily observed that  $V(y) = -\log \varphi(y)$  satisfies

$$V'(0) - V'(y) = \frac{\int u m(du)}{\int m(du)} - \frac{\int e^{-uy} u m(du)}{\int e^{-uy} m(du)} \leq \frac{\overline{m}_1}{\overline{m}_0} - \inf \mathcal{S}(m)$$

for all  $y > 0$ . On the other hand, it follows from (4.10) that

$$\left\| \frac{d\tilde{n}}{dn} \right\|_\infty \leq \frac{1}{\inf \mathcal{S}(M)} \leq \frac{1}{\inf \mathcal{S}(m)},$$

where the last inequality is implied by Lemma 4.2 (iv). Therefore, (4.11) is deduced from (3.18).  $\blacksquare$

Notice that the upper estimates (4.11) are effective in the sense that the most right side vanishes only in the reversible case, namely when  $m$  is degenerate. The symmetry found in the statement of Lemma 4.1 allows one to show a converse of Theorem 4.3. Roughly speaking, we will see below that every ergodic CBCI-process with  $n$  having a (non-zero) completely monotone density has some GGC as a (non-reversible) stationary distribution.

**Theorem 4.4** *Let  $a, b \geq 0$  and suppose that a non-zero  $M \in \mathcal{M}$  is given. Define  $n$  and  $\Phi$  by (4.10) and the first equality in (3.3), respectively. Then  $\int \min\{y, y^2\}n(dy)$  is finite and the following assertions hold true.*

(i)  $\Phi(1) < \infty$  if and only if  $b > 0$  or

$$\int_0^1 \left( u \int \frac{M(dv)}{u+v} \right)^{-1} du < \infty.$$

(ii) If  $\Phi(1) < \infty$ , then the unique stationary distribution of the CBCI-process with quadruplet  $(a, b, n, 1)$  is a GGC with some pair  $(q, m)$  such that

$$q = \begin{cases} 0 & (a > 0) \\ 1/(b + \overline{M}_0) & (a = 0), \end{cases} \quad \overline{m}_0 = \begin{cases} 1/a & (a > 0) \\ \overline{M}_1/(b + \overline{M}_0)^2 & (a = 0, \overline{M}_0 < \infty) \\ \infty & (a = 0, \overline{M}_0 = \infty). \end{cases} \quad (4.13)$$

(iii) If  $a, b > 0, \overline{M}_0 < \infty$  and  $\inf \mathcal{S}(M) > 0$ , then for any  $\delta > 0$  the CBCI-process with quadruplet  $(a, b, n, \delta)$  is ergodic and  $\left( \text{Sect}(\mathcal{E}^\delta) - 1 \right)^2$  is dominated by

$$\frac{\sqrt{\left( a \inf \mathcal{S}(M) - b - \overline{M}_0 \right)^2 + 4a\overline{M}_0 \inf \mathcal{S}(M) - \left( a \inf \mathcal{S}(M) - b - \overline{M}_0 \right)}}{a \inf \mathcal{S}(M)}. \quad (4.14)$$

*Proof.* By virtue of (4.12), that  $\int \min\{y, y^2\}n(dy) < \infty$  is ensured by  $M \in \mathcal{M}$ . Putting  $g_M(u) = \int (u+v)^{-1}M(dv)$ , note that  $\Phi(\lambda) = \int_0^\lambda (au + b + ug_M(u))^{-1}du$ . For the proof of (i), it is sufficient to show that  $b = 0$  and  $\int_0^1 (ug_M(u))^{-1}du = \infty$  together imply  $\Phi(1) = \infty$ . If  $\overline{M}_{-1} < \infty$ , this holds true since  $\Phi(1) \geq \int_0^1 (au + u\overline{M}_{-1})^{-1}du = \infty$ . If  $\overline{M}_{-1} = \infty$ , there exists  $v \in (0, 1)$  such that  $a < g_M(u)$  for any  $u \in (0, v]$  and hence  $\Phi(1) \geq \int_0^v (2ug_M(u))^{-1}du = \infty$ . These observations prove (i).

To show (ii) we interchange in Lemma 4.1 the roles of  $(a, b, M)$  and  $(q, r, m)$  to get

$$a + \frac{b}{\lambda} + \int \frac{1}{\lambda + u} M(du) = \frac{1}{q\lambda + r + \int \frac{\lambda}{\lambda + u} m(du)}, \quad \lambda > 0 \quad (4.15)$$

for some  $q, r \geq 0$  and  $m \in \mathcal{M}$ . Here, according to (4.6),  $q$  is given by (4.13),  $r = 0$  if  $b > 0$ , and  $r = 1/(a + \overline{M}_{-1})$  if  $b = 0$ . But it follows from  $\Phi(1) < \infty$  and (i) that

$\overline{M}_{-1} = \infty$  whenever  $b = 0$ . As a result  $r = 0$ , and therefore the formula (4.13) for  $\overline{m}_0$  precisely corresponds to (4.7). By (4.15) and Fubini's theorem one can show that  $\Phi(\lambda) = \Phi_{q,m}(\lambda)$ . Since  $\Phi(1) < \infty$ , this proves not only that  $m$  is a Thorin measure but also the assertion (ii).

It remains to show the sector constant estimate under the stronger assumptions that  $a, b > 0, \overline{M}_0 < \infty$  and  $\inf \mathcal{S}(M) > 0$ . This should be derived from (4.11) once  $\overline{m}_1/\overline{m}_0$  and  $\inf \mathcal{S}(m)$  are estimated in terms of  $a, b$  and  $M$ . For this purpose, combine  $\overline{m}_0 = 1/a$  with  $\overline{M}_0 = \overline{m}_1/\overline{m}_0^2 - b$  to get  $\overline{m}_1/\overline{m}_0 = (\overline{M}_0 + b)/a$ . Also, Lemma 4.2 (i) and (4.9) with  $(q, r, M)$  being interchanged with  $(a, b, m)$  are applied to derive

$$\begin{aligned} \inf \mathcal{S}(M) &\geq s_-(a, b, M) \geq s'_-(a, b, M) \\ &= \frac{\left(a \inf \mathcal{S}(M) + b + \overline{M}_0\right) - \sqrt{\left(a \inf \mathcal{S}(M) + b + \overline{M}_0\right)^2 - 4ab \inf \mathcal{S}(M)}}{2a}. \end{aligned}$$

These calculations together yield

$$\begin{aligned} \frac{\overline{m}_1}{\overline{m}_0} - \inf \mathcal{S}(m) &\leq \frac{\left(b + \overline{M}_0\right) - a \inf \mathcal{S}(M) + \sqrt{\left(a \inf \mathcal{S}(M) + b + \overline{M}_0\right)^2 - 4ab \inf \mathcal{S}(M)}}{2a} \\ &= \frac{\sqrt{\left(a \inf \mathcal{S}(M) - b - \overline{M}_0\right)^2 + 4a\overline{M}_0 \inf \mathcal{S}(M)} - \left(a \inf \mathcal{S}(M) - b - \overline{M}_0\right)}{2a}. \end{aligned}$$

Therefore, the bound (4.14) is deduced from the first inequality in (4.11).  $\blacksquare$

To check, consider the case discussed in Example 2.1 (ii). It is easily seen that the Laplace exponent (2.23) is that of the GGC with  $q = 0$  and Thorin measure

$$m(du) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{\mathbf{1}_{(\kappa, \infty)}(u)}{(u - \kappa)^\alpha} du. \quad (4.16)$$

Accordingly,  $\overline{m}_0 = \infty$  and  $\overline{m}_{-1} = \kappa^{-\alpha}$ . So, (4.6) gives  $a = 0$  and  $b = \kappa^\alpha$  consistently. Although Lemma 4.1 itself does not give any explicit form of  $M$ , we can identify it with

$$M(du) = \frac{u^{-1}(u - \kappa)^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \mathbf{1}_{(\kappa, \infty)}(u) du, \quad (4.17)$$

thanks to (2.22). Therefore, in this case  $\mathcal{S}(M) = \mathcal{S}(m) = (\kappa, \infty)$ . In the next section we will be provided with some procedure to derive a formulae for  $M$  including (4.17) via (4.5) and with further examples as well.

As for the lower bound of the sector constant of CBCI-processes discussed in Theorem 4.3, one can show the next result as an application of Proposition 3.7.

**Theorem 4.5** *Suppose that  $m$  is a non-zero measure on  $(0, \infty)$  with  $\inf \mathcal{S}(m) > 0$  and  $\overline{m}_0 < \infty$ . Let  $a, b > 0$  and  $M \in \mathcal{M}$  be as in Lemma 4.1 with  $q = 0 = r$  and  $n$*



be given by (4.10). Then the bilinear form  $\mathcal{E}$  associated with the CBCI-process with quadruplet  $(a, b, n, 1)$  satisfies

$$\text{Sect}(\mathcal{E})^2 - 1 \geq \frac{(\overline{m}_{-1}\overline{m}_{-3} - \overline{m}_{-2}^2)^2}{2\overline{m}_{-1}\overline{m}_{-2}\overline{m}_{-3} + 4\overline{m}_{-1}^2\overline{m}_{-2}^3 + 12\overline{m}_{-1}^2\overline{m}_{-2}\overline{m}_{-4} - 9\overline{m}_{-1}^2\overline{m}_{-3}^2 - \overline{m}_{-2}^4}.$$

*Proof* By the assumption  $m$  is a Thorin measure and the GGC with pair  $(0, m)$  has an exponential integrability. Indeed, its Laplace exponent  $\Phi_{0,m}(\lambda)$  can extend real analytically to  $\lambda > -\inf \mathcal{S}(m)$ . In particular, we have the finite moments given by

$$\langle x^k \rangle = (-1)^k \left. \frac{d^k}{d\lambda^k} e^{-\Phi_{0,m}(\lambda)} \right|_{\lambda=0}$$

for  $k = 1, 2, \dots$ . Since  $\Phi_{0,m}^{(k)}(0) = (-1)^{k-1}(k-1)!\overline{m}_{-k}$ , it follows that

$$\langle x \rangle = \overline{m}_{-1}, \quad \langle x^2 \rangle = \overline{m}_{-1}^2 + \overline{m}_{-2} \text{ and } \langle x^3 \rangle = 2\overline{m}_{-3} + 3\overline{m}_{-1}\overline{m}_{-2} + \overline{m}_{-1}^3. \quad (4.18)$$

We are going to apply (3.19) by taking  $f(x) = x$  and  $g(x) = x^2$ , for which  $\Delta(f, g) > 0$  holds by virtue of (3.2) and Schwarz's inequality together. Explicitly, our task is to show for some common  $C > 0$  that

$$\tilde{\mathcal{E}}(f, g)^2 = C(\overline{m}_{-1}\overline{m}_{-3} - \overline{m}_{-2}^2)^2 \quad (4.19)$$

and that  $\Delta(f, g) = \mathcal{E}(f, f)\mathcal{E}(g, g) - \tilde{\mathcal{E}}(f, g)^2$  equals

$$C(2\overline{m}_{-1}\overline{m}_{-2}^2\overline{m}_{-3} + 4\overline{m}_{-1}^2\overline{m}_{-2}^3 + 12\overline{m}_{-1}^2\overline{m}_{-2}\overline{m}_{-4} - 9\overline{m}_{-1}^2\overline{m}_{-3}^2 - \overline{m}_{-2}^4). \quad (4.20)$$

Noting that  $\int n(dy)y^k = k!\overline{M}_{1-k}$  ( $k = 1, 2, \dots$ ) by (4.10), observe from (an extension of) (3.5) and (3.2) that

$$\tilde{\mathcal{E}}(f, g) = \frac{1}{2}\langle x \int \tilde{n}(dy)(-y^2) \rangle = -\frac{1}{6}\langle x \rangle \int n(dy)y^3 = -\langle x \rangle \overline{M}_{-2}, \quad (4.21)$$

$$\mathcal{E}(f, f) = \langle x \rangle \left( a + \frac{1}{2} \int n(dy)y^2 \right) = \langle x \rangle (a + \overline{M}_{-1}),$$

$$\begin{aligned} \tilde{\mathcal{E}}(f, g) &= 2\langle x^2 \rangle \left( a + \frac{1}{2} \int n(dy)y^2 \right) + \frac{1}{2}\langle x \rangle \int n(dy)y^3 \\ &= 2\langle x^2 \rangle (a + \overline{M}_{-1}) + 3\langle x \rangle \overline{M}_{-2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}(g, g) &= 4\langle x^3 \rangle \left( a + \frac{1}{2} \int n(dy)y^2 \right) + 2\langle x^2 \rangle \int n(dy)y^3 + \frac{1}{2}\langle x \rangle \int n(dy)y^4 \\ &= 4\langle x^3 \rangle (a + \overline{M}_{-1}) + 12\langle x^2 \rangle \overline{M}_{-2} + 12\langle x \rangle \overline{M}_{-3}. \end{aligned}$$

The last three equalities together yield

$$\begin{aligned}\Delta(f, g) &= 4 \left( \langle x \rangle \langle x^3 \rangle - \langle x^2 \rangle^2 \right) \left( a + \overline{M}_{-1} \right)^2 \\ &\quad + 12 \langle x \rangle^2 \left( a + \overline{M}_{-1} \right) \overline{M}_{-3} - 9 \langle x \rangle^2 \overline{M}_{-2}^2.\end{aligned}\quad (4.22)$$

We shall calculate  $\overline{M}_{-1}$ ,  $\overline{M}_{-2}$  and  $\overline{M}_{-3}$ . The results are

$$\overline{M}_{-1} = \frac{\overline{m}_0 \overline{m}_{-2} - \overline{m}_{-1}^2}{\overline{m}_0 \overline{m}_{-1}^2}, \quad \overline{M}_{-2} = \frac{\overline{m}_{-1} \overline{m}_{-3} - \overline{m}_{-2}^2}{\overline{m}_{-1}^3} \quad (4.23)$$

and

$$\overline{M}_{-3} = \frac{\overline{m}_{-1}^2 \overline{m}_{-4} - 2 \overline{m}_{-1} \overline{m}_{-2} \overline{m}_{-3} + \overline{m}_{-2}^3}{\overline{m}_{-1}^4}, \quad (4.24)$$

whose proof are postponed for a while. Then (4.19) with  $C = 1/\overline{m}_{-1}^4$  follows immediately by plugging (4.18) and (4.23) into (4.21). In principle, (4.20) with the same  $C$  can be obtained similarly though the calculation is tedious. To carry this out, observe that  $a + \overline{M}_{-1} = \overline{m}_{-2}/\overline{m}_{-1}^2$  by  $a = 1/\overline{m}_0$  and that (4.18) give

$$\langle x \rangle \langle x^3 \rangle - \langle x^2 \rangle^2 = \overline{m}_{-1}^2 \overline{m}_{-2} + 2 \overline{m}_{-1} \overline{m}_{-3} - \overline{m}_{-2}^2.$$

(4.20) with  $C = 1/\overline{m}_{-1}^4$  will now be derived in a fairly straightforward way.

It remains to prove (4.23) and (4.24). We exploit a variant of the identity used to find  $\overline{M}_0$  in the proof of Lemma 4.1 (with  $q = 0 = r$ ):

$$\int \frac{1}{\lambda + u} M(du) = \frac{\int \frac{u}{\lambda + u} m(du)}{\overline{m}_0 \int \frac{\lambda}{\lambda + u} m(du)} - \frac{1}{\overline{m}_{-1} \lambda}, \quad \lambda > 0. \quad (4.25)$$

Letting  $\lambda \downarrow 0$ , we get (4.23) for  $\overline{M}_{-1}$  with the help of L'Hospital's rule. By differentiating (4.25)

$$\begin{aligned}- \int \frac{1}{(\lambda + u)^2} M(du) &= \frac{1}{\lambda^2} \cdot \frac{- \int \frac{u m(du)}{(\lambda + u)^2} \int \frac{m(dv)}{v} + \left( \int \frac{m(du)}{\lambda + u} \right)^2}{\overline{m}_{-1} \left( \int \frac{m(du)}{\lambda + u} \right)^2} \\ &= - \frac{\int \int \frac{(u - v)^2}{(\lambda + u)^2 u v (\lambda + v)^2} m(du) m(dv)}{2 \overline{m}_{-1} \left( \int \frac{m(du)}{\lambda + u} \right)^2},\end{aligned}\quad (4.26)$$

where the symmetry of  $m(du)m(dv)$  applies to show the last equality. Letting  $\lambda \downarrow 0$  leads to (4.23) for  $\overline{M}_{-2}$ . Finally, differentiating the both sides of (4.26) multiplied by the square of  $\int (\lambda + u)^{-1} m(du)$  and then letting  $\lambda \downarrow 0$ , we see without difficulty that

$$2 \overline{M}_{-3} \overline{m}_{-1}^2 + 2 \overline{M}_{-2} \overline{m}_{-1} \overline{m}_{-2} = \frac{2(\overline{m}_{-1} \overline{m}_{-4} - \overline{m}_{-2} \overline{m}_{-3})}{\overline{m}_{-1}}.$$

This combined with (4.23) proves (4.24). The proof of Theorem 4.5 is complete.  $\blacksquare$

## 5 Further discussions and related topics

In this section, most calculations will be based on the equations (4.5) and (4.8) with  $r = 0$ , and therefore we take  $r = 0$  without explicit mention (except in the statements of Propositions 5.2 and 5.3 below). The first two subsections are devoted to further studies of the correspondence between  $(a, b, M)$  and  $(q, m)$  under some special circumstances. In the final subsection, the basic relation (4.5) will be discussed also in connection with certain topics in noncommutative probability theory.

### 5.1 Discrete Thorin measures

This subsection concerns the correspondence between GGC's and ergodic CBCI processes under the condition that  $m$  (or  $M$ ) is discrete. As the simplest example of such GGC's (other than gamma distributions) we first discuss the case of convolutions of two gamma distributions in rather detail. Given  $x \in \mathbf{R}$ , denote by  $\epsilon_x$  the delta distribution at  $x$ .

*Example 5.1.* (i) Let  $\gamma_1, \gamma_2, \lambda_1, \lambda_2$  be positive constants. Consider  $m := \gamma_1 \epsilon_{\lambda_1} + \gamma_2 \epsilon_{\lambda_2}$  as the Thorin measure. According to (4.6) with  $q = 0$ ,  $a$  and  $b$  are chosen as

$$a = \frac{1}{\gamma_1 + \gamma_2} \quad \text{and} \quad b = \frac{1}{\lambda_1^{-1} \gamma_1 + \lambda_2^{-1} \gamma_2}, \quad (5.1)$$

respectively. Our ansatz here is that the measure  $M$  satisfying (4.5) with  $q = 0$  is of the form  $M = c \epsilon_\kappa$  for some  $c, \kappa > 0$ , which are to be determined. The measure  $n$  in (4.10) is then given by  $n(dy) = c \kappa^2 e^{-\kappa y} dy$  and the equation (4.5) reads

$$\frac{\gamma_1}{\lambda + \lambda_1} + \frac{\gamma_2}{\lambda + \lambda_2} = \frac{1}{a\lambda + b + \frac{c\lambda}{\lambda + \kappa}}, \quad \lambda \geq 0. \quad (5.2)$$

It is not difficult to see that the above requirement is fulfilled by setting

$$\kappa := \frac{a}{b} \lambda_1 \lambda_2 = \frac{\lambda_2 \gamma_1 + \lambda_1 \gamma_2}{\gamma_1 + \gamma_2} \quad (5.3)$$

and

$$c := a(\lambda_1 + \lambda_2 - \kappa) - b = \frac{\gamma_1 \gamma_2 (\lambda_1 - \lambda_2)^2}{(\gamma_1 + \gamma_2)^2 (\lambda_2 \gamma_1 + \lambda_1 \gamma_2)}, \quad (5.4)$$

which vanishes for degenerate  $m$  with  $\lambda_1 = \lambda_2$  in accordance with the comments in the paragraph preceding to Lemma 4.2. Lastly, assuming  $\lambda_1 \leq \lambda_2$  and letting  $\delta > 0$ , we have by the first bound in (4.11) combined with (5.3)

$$\text{Sect}(\mathcal{E}^\delta) - 1 \leq \sqrt{\left( \frac{\gamma_1 \lambda_1 + \gamma_2 \lambda_2}{\gamma_1 + \gamma_2} - \lambda_1 \right) \frac{2}{\kappa}} = \sqrt{\frac{2(\lambda_2 - \lambda_1) \gamma_2}{\lambda_2 \gamma_1 + \lambda_1 \gamma_2}} \quad (5.5)$$

for the CBCI-process with quadruplet  $(a, b, n, \delta)$ .

(ii) One can reverse the above procedure to construct explicitly the Lévy density of the stationary distribution of the CBCI-process with  $n$  being of the form  $n(dy) = c\kappa^2 e^{-\kappa y} dy$ . Indeed, given  $a, b, c, \kappa > 0$ , we find  $\lambda_1, \lambda_2, \gamma_1, \gamma_2$  satisfying (5.2) in the following manner. In view of the first equalities of (5.3) and (5.4), the required  $\lambda_i$ 's must solve the equation  $p(\lambda) := a\lambda^2 - (a\kappa + b + c)\lambda + b\kappa = 0$ . This leads to

$$\lambda_1 = \frac{(a\kappa + b + c) - \sqrt{D}}{2a} \quad \text{and} \quad \lambda_2 = \frac{(a\kappa + b + c) + \sqrt{D}}{2a},$$

where  $D = (a\kappa + b + c)^2 - 4ab\kappa = (a\kappa - b - c)^2 + 4ack > 0$ . Note that  $\lambda_1 < \kappa < \lambda_2$  since  $p(\kappa) < 0$ . Moreover,  $\gamma_i$ 's are determined by

$$\gamma_1 = \frac{\kappa - \lambda_1}{a(\lambda_2 - \lambda_1)} \quad \text{and} \quad \gamma_2 = \frac{\lambda_2 - \kappa}{a(\lambda_2 - \lambda_1)},$$

for which (5.1) are easily checked to hold. Consequently, the Lévy density (4.1) equals  $(\gamma_1 e^{-\lambda_1 y} + \gamma_2 e^{-\lambda_2 y})/y$ . In addition, one can derive from (5.5) the sector constant estimate described in terms of  $a, b, c$  and  $\kappa$  by noting that

$$\frac{2(\lambda_2 - \lambda_1)\gamma_2}{\lambda_2\gamma_1 + \lambda_1\gamma_2} = \frac{2(\lambda_2 - \kappa)}{\kappa} = \frac{\sqrt{D} - (a\kappa - b - c)}{a\kappa},$$

which clearly corresponds to (4.14).

*Example 5.2.* Given  $q > 0$  and a degenerate Thorin measure  $m$ , we have a similar situation to Example 5.1 except the sector constant estimate, which is not available since  $a = 0$  by (4.6). Indeed, for  $m = \gamma_1 \epsilon_{\lambda_1}$  with  $\gamma_1, \lambda_1 > 0$  being fixed arbitrarily,  $b = 1/(q + \gamma_1/\lambda_1)$  by (4.6) and it is easily verified that the measure  $M$  satisfying (4.5) is given by

$$M = \left( \frac{1}{q} - \frac{1}{q + \gamma_1/\lambda_1} \right) \epsilon_{\lambda_1 + \gamma_1/q}.$$

Conversely, if we are given  $b > 0$  and  $M = c\epsilon_\kappa$  with  $c, \kappa > 0$ , then the measure  $m$  determined by (4.5) with  $q = 1/(b + c)$  and  $a = 0$  is shown to be

$$m = \frac{c\kappa}{(b + c)^2} \epsilon_{b\kappa/(b+c)}.$$

Alternatively, this can be derived directly from (3.4) by noting that  $\tilde{n}^{*N}(dy) = (c\kappa)^N y^{N-1} e^{-\kappa y} dy / (N-1)!$  for  $N = 1, 2, \dots$ .

Apart from explicit expressions, the above examples are generalized as follows.

**Proposition 5.1** *Let  $l \geq 1$  be a fixed integer.*

(i) *For every discrete measure*

$$m = \gamma_1 \epsilon_{\lambda_1} + \dots + \gamma_{l+1} \epsilon_{\lambda_{l+1}} \quad \text{with} \quad \gamma_i > 0 \text{ and } 0 < \lambda_1 < \dots < \lambda_{l+1}, \quad (5.6)$$

*there exists a unique measure  $M$  of the form*

$$M = c_1 \epsilon_{\kappa_1} + \dots + c_l \epsilon_{\kappa_l} \quad \text{with} \quad c_i > 0 \text{ and } 0 < \kappa_1 < \dots < \kappa_l \quad (5.7)$$

satisfying (4.5) with  $a = 1/\overline{m}_0$ ,  $b = 1/\overline{m}_{-1}$  and  $q = 0 = r$ . Furthermore, the sequences  $\{\lambda_i\}$  and  $\{\kappa_i\}$  interlace:

$$\lambda_1 < \kappa_1 < \lambda_2 < \cdots < \kappa_{l-1} < \lambda_l < \kappa_l < \lambda_{l+1}. \quad (5.8)$$

Conversely, for given  $a, b > 0$  and a discrete measure (5.7), there exists a unique measure  $m$  of the form (5.6) satisfying (4.5) with  $q = 0 = r$ .

(ii) Let  $q > 0$  be given. For every discrete measure

$$m = \gamma_1 \epsilon_{\lambda_1} + \cdots + \gamma_l \epsilon_{\lambda_l} \quad \text{with} \quad \gamma_i > 0 \text{ and } 0 < \lambda_1 < \cdots < \lambda_l, \quad (5.9)$$

there exists a unique measure  $M$  of the form (5.7) satisfying (4.5) with  $r = 0 = a$  and  $b = 1/(q + \overline{m}_{-1})$ . Furthermore, it holds that

$$\lambda_1 < \kappa_1 < \lambda_2 < \cdots < \kappa_{l-1} < \lambda_l < \kappa_l \leq \lambda_l + \overline{m}_0 q^{-1}. \quad (5.10)$$

Conversely, for given  $b > 0$  and a discrete measure (5.7), there exists a unique measure  $m$  of the form (5.9) satisfying (4.5) with  $q = 1/(b + \overline{M}_0)$  and  $r = 0 = a$ .

*Proof.* (i) Observe from (4.5) with  $q = 0$  that  $g(\lambda) := f(\lambda + u)^{-1} M(du)$  is a rational function of the form

$$g(\lambda) = \frac{1}{\lambda} \left( \frac{P(\lambda)}{Q(\lambda)} - (a\lambda + b) \right) = \frac{P(\lambda) - (a\lambda + b)Q(\lambda)}{\lambda} \cdot \frac{1}{Q(\lambda)},$$

where  $a$  and  $b$  are given by (4.6) and

$$P(\lambda) = \prod_{i=1}^{l+1} (\lambda + \lambda_i), \quad Q(\lambda) = \sum_{i=1}^l \gamma_i \prod_{j \neq i} (\lambda + \lambda_j).$$

We see also that  $P_0(\lambda) := (P(\lambda) - (a\lambda + b)Q(\lambda))/\lambda$  is in fact a polynomial with degree less than or equal to  $l - 1$ , and that  $Q$  has a zero in the interval  $(-\lambda_{i+1}, -\lambda_i)$  for each  $i \in \{1, \dots, l\}$  because  $Q(-\lambda_i)/Q(-\lambda_{i+1}) < 0$  as observed from (5.6). Therefore,  $g(\lambda) = aP_0(\lambda)/((\lambda + \kappa_1) \cdots (\lambda + \kappa_l))$  for some  $\kappa_1, \dots, \kappa_l$  satisfying (5.8). It remains to find  $c_1, \dots, c_l > 0$  such that

$$aP_0(\lambda) = \sum_{i=1}^l c_i \prod_{j \neq i} (\lambda + \kappa_j), \quad \lambda > 0.$$

Since  $\kappa_1, \dots, \kappa_l$  are mutually different, a necessary and sufficient condition for the above identity to hold is that

$$aP_0(-\kappa_i) = c_i \prod_{j \neq i} (-\kappa_i + \kappa_j), \quad i \in \{1, \dots, l\}, \quad (5.11)$$

which uniquely determine  $c_1, \dots, c_l$ . Noting that  $P_0(-\kappa_i) = P(-\kappa_i)/(-\kappa_i)$  because of  $Q(-\kappa_i) = 0$ , we can verify the positivity of  $c_i$ 's by making use of (5.11) and (5.8).

It is almost a routine matter to show the converse assertion, whose proof we shall sketch. Let  $M$  be given by (5.7). Defining this time

$$P(\lambda) = \prod_{i=1}^l (\lambda + \kappa_i) \quad \text{and} \quad Q(\lambda) = \sum_{i=1}^l c_i \prod_{j \neq i} (\lambda + \kappa_j),$$

we only have to show that the rational function

$$\left( (a\lambda + b) + \lambda \frac{Q(\lambda)}{P(\lambda)} \right)^{-1} =: \frac{P(\lambda)}{Q_0(\lambda)}$$

can be rewritten into  $\sum_{i=1}^{l+1} \gamma_i / (\lambda + \lambda_i)$  for some  $\gamma_i > 0$  and  $\lambda_i > 0$  satisfying (5.8). Such  $\lambda_2, \dots, \lambda_l$  are found as zeros of  $Q_0$  since  $Q_0(-\kappa_i)/Q_0(-\kappa_{i+1}) < 0$  ( $i = 1, \dots, l-1$ ). Therefore

$$Q_0(\lambda) = (\lambda + \lambda_2) \cdots (\lambda + \lambda_l) Q_1(\lambda) \quad (5.12)$$

for some quadratic polynomial  $Q_1(\lambda) = a\lambda^2 + q_1\lambda + q_2$  with  $q_1 \in \mathbf{R}$  and  $q_2 > 0$ . Moreover, with the help of (5.12) and (5.8), we can show that  $Q_1(-\kappa_i) < 0$  for each  $i \in \{1, \dots, l\}$ . These observations imply that  $Q_1(\lambda) = a(\lambda + \lambda_1)(\lambda + \lambda_l)$  for some  $\lambda_1 \in (0, \kappa_1)$  and  $\lambda_{l+1} \in (\kappa_l, \infty)$ . The rest of the proof (i.e. finding  $\gamma_i$ 's) is the same as before and the details are left to the reader.

The assertion (ii) can be shown in an analogous way to the assertion (i). So we omit the proof of (ii) except the last inequality in (5.10), which follows immediately from Lemma 4.2 (iii).  $\blacksquare$

## 5.2 Absolutely continuous Thorin measures

As for continuous  $m$ , combining (4.8) with the following inversion formula of Stieltjes-Perron (cf. [24], p.340, Corollary 7a) may provide explicit information on  $M$ :

$$M((s, t)) + \frac{M(\{s\}) + M(\{t\})}{2} = -\frac{1}{\pi} \lim_{y \downarrow 0} \int_s^t \operatorname{Im} G_M(x + iy) dx, \quad -\infty < s < t < \infty,$$

where  $i = \sqrt{-1}$ . In particular,  $M$  has a density function given by

$$\frac{dM}{dx} = -\frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im} G_M(x + iy), \quad x \in [s, t],$$

provided that the right side converges boundedly and pointwise on  $[s, t]$ . We try to evaluate the right side in our setting. Let  $m \in \mathcal{M}$  be non-zero and set  $q = 0$  for simplicity. Rewrite (4.8) in the form  $G_M(z) = -1/(zG_m(z)) + a - b/z$ , where  $a, b \geq 0$  and  $M \in \mathcal{M}$  correspond to the pair  $(0, m)$  in the sense of Theorem 4.3. In the case where  $m$  is absolutely continuous, it would be possible to establish a density formula for  $M$  under suitable conditions, which do not seem, however, to be prescribed in a neat fashion. In typical cases, there exists a signed measure  $m'$  on  $[0, \infty)$  such that  $G_m(z) = \int_{[0, \infty)} \log(z - u) m'(du)$  and hence

$$\lim_{y \downarrow 0} G_m(x + iy) = \int_{[0, \infty)} \log|x - u| m'(du) + i\pi m'((x, \infty))$$

for each  $x > 0$  with  $m'(\{x\}) = 0$ . Therefore, the density formula would take the form

$$\begin{aligned} \frac{dM}{dx} &= \frac{1}{\pi x} \lim_{y \downarrow 0} \operatorname{Im} \frac{1}{G_m(x + iy)} \\ &= -\frac{1}{x} \cdot \frac{m'((x, \infty))}{\left( \int_{[0, \infty)} \log |x - u| m'(du) \right)^2 + (\pi m'((x, \infty)))^2} \end{aligned} \quad (5.13)$$

for  $x > 0$ . Similarly, (4.8) in principle makes it possible to derive the information of  $m$  corresponding to given  $a, b \geq 0$  and  $M \in \mathcal{M}$ . We shall give some concrete examples of such a kind, in which we continue to take  $q = 0$ .

*Example 5.3* (i) For  $0 < \alpha < 1$  and  $\kappa \geq 0$ , let  $m$  be as in (4.16). Then  $G_m(z) = -(\kappa - z)^{-\alpha}$ . Here the power function  $z^\alpha$  of  $z \in \mathbf{C} \setminus (-\infty, 0]$  is defined to be  $|z|^\alpha (\cos \arg z + i \sin \arg z)$  with  $\arg z$  chosen so that  $|\arg z| < \pi$ . Since we know from Lemma 4.2 (iii) that  $\mathcal{S}(M) \subset [\kappa, \infty)$ , fix an  $x > \kappa$  arbitrarily. It is easy to see that

$$\frac{1}{\pi x} \lim_{y \downarrow 0} \operatorname{Im} \frac{1}{G_m(x + iy)} = -\frac{1}{\pi x} (x - \kappa)^\alpha \sin(\alpha(-\pi)) = \frac{1}{x} \cdot \frac{(x - \kappa)^\alpha}{\Gamma(\alpha)\Gamma(1 - \alpha)}.$$

It is not difficult to verify that the above convergence is uniform in  $x$  on every compact interval contained in  $(\kappa, \infty)$ . Thus (4.17) has been recovered.

(ii) Let  $0 \leq \lambda_1 < \lambda_2$  be arbitrary. Define  $m(du) = \mathbf{1}_{(\lambda_1, \lambda_2)}(u)du$ , for which  $a = 1/(\lambda_1 - \lambda_2)$ ,  $b = 1/(\log \lambda_2 - \log \lambda_1)$ . Moreover,  $G_m(z) = \log(z - \lambda_1) - \log(z - \lambda_2)$  and hence  $m' = \epsilon_{\lambda_1} - \epsilon_{\lambda_2}$ . By (5.13)

$$\frac{dM}{dx} = \frac{1}{x} \cdot \frac{\mathbf{1}_{(\lambda_1, \lambda_2)}(x)}{\left( \log \frac{x - \lambda_1}{\lambda_2 - x} \right)^2 + \pi^2}. \quad (5.14)$$

While (4.7) tells that

$$M([\lambda_1, \lambda_2]) = \frac{\lambda_1 + \lambda_2}{2(\lambda_2 - \lambda_1)} - \frac{1}{\log \lambda_2 - \log \lambda_1},$$

it is not clear how to verify this directly from (5.14) unless  $\lambda_1 = 0$ . If  $\lambda_1 > 0$ , we have also the sector constant estimate (4.11) which reads  $\operatorname{Sect}(\mathcal{E}^\delta) - 1 \leq \sqrt{(\lambda_2 - \lambda_1)/\lambda_1}$  for each  $\delta > 0$ . For the special choice  $\lambda_1 = 0$  and  $\lambda_2 = 1$ , the density function of the GGC with pair  $(0, m)$  can be found in [6] (Eq.(259)).

(iii) Let  $a, b > 0$  be given arbitrarily. Consider  $M(du) = \mathbf{1}_{(0, 1)}(u)du$  for simplicity. The density of the absolutely continuous part  $m_c$  of the Thorin measure  $m$  corresponding to  $(a, b, M)$  in the sense of Theorem 4.4 (ii) can be calculated from (4.8) with  $q = 0 = r$ , namely  $G_m(z) = 1/(az - b - zG_M(z))$ , where  $G_M(z) = \log z - \log(z - 1)$ . Indeed, defining  $H(z) = a - (b/z) - G_M(z)$  for  $z \neq 0$ , one can show that, for each

$x > 0$  with  $x \neq 1$ ,  $H(x + iy) \rightarrow a - (b/x) - \log |x/(x-1)| + i\pi \mathbf{1}_{(0,1)}(x)$  as  $y \downarrow 0$ . Therefore,

$$\frac{dm_c}{dx} = -\frac{1}{\pi x} \lim_{y \downarrow 0} \operatorname{Im} \frac{1}{H(x + iy)} = \frac{1}{x} \cdot \frac{\mathbf{1}_{(0,1)}(x)}{\left(a - \frac{b}{x} - \log \frac{x}{1-x}\right)^2 + \pi^2}.$$

The point which requires extra care is the unique pole, say  $x_0$ , of  $G_m$ , which is located on the interval  $(1, \infty)$ . It is characterized as a unique solution to

$$ax - b + x \log \left(1 - \frac{1}{x}\right) = 0, \quad (5.15)$$

from which  $b/a < x_0 < 1 + (b+1)/a$  can be deduced with the help of elementary inequalities  $u/(1+u) < \log(1+u) < u$  for  $u > -1$ . The point mass of  $m$  at  $x_0$  is given as the residue of  $G_m$  at  $z = x_0$ :

$$\begin{aligned} m(\{x_0\}) &= \lim_{z \rightarrow x_0} G_m(z)(z - x_0) = \left( \frac{d}{dz} (az - b - zG_m(z)) \Big|_{z=x_0} \right)^{-1} \\ &= \left( a + \log \left(1 - \frac{1}{x_0}\right) + \frac{1}{x_0} \right)^{-1} = \left( \frac{b}{x_0} + \frac{1}{x_0 - 1} \right)^{-1}, \end{aligned}$$

where (5.15) has been used to get the last expression.

### 5.3 Connections with noncommutative probability theory

In the previous subsection, the calculus of Stieltjes transforms played quite an important role. So, it might be no surprise that observations we had made so far have some connections with noncommutative probability theory, another context in which the reciprocal of the Stieltjes transform serves as one of essential tools. We will be particularly concerned with the Boolean convolution and the free Poisson distributions (known also as the Marchenko-Pasture laws). To give the definition of the former, we follow [19] and introduce the ‘Boolean cumulant’  $K_m(z) := z - 1/G_m(z)$  for a *probability* measure  $m$  on  $\mathbf{R}$ . For our purpose the domain of such an operation shall be restricted to  $\mathcal{M}_1$ , the totality of probability measures on  $(0, \infty)$ . For any  $m_1, m_2 \in \mathcal{M}_1$ , the Boolean convolution  $m_1 \uplus m_2$  of  $m_1$  and  $m_2$  is then defined to be an element of  $\mathcal{M}_1$  such that

$$K_{m_1 \uplus m_2}(z) = K_{m_1}(z) + K_{m_2}(z), \quad z \in \mathbf{C} \setminus \mathbf{R}_+. \quad (5.16)$$

Let  $t > 0$  be arbitrary. Following [3], one can define also the  $t$ th Boolean convolution power  $m^{\uplus t}$  of  $m \in \mathcal{M}_1$  by

$$K_{m^{\uplus t}}(z) = t \cdot K_m(z), \quad z \in \mathbf{C} \setminus \mathbf{R}_+. \quad (5.17)$$

It is a good exercise to verify from Lemma 4.1 with  $a = 1$  that the requirements (5.16) and (5.17) determine uniquely such measures  $m_1 \uplus m_2$  and  $m^{\uplus t}$ , respectively. (In fact,



for the verification, one needs to observe in Lemma 4.1 additionally that  $\overline{M}_{-1} = \infty$  whenever  $b = 0$ . But this can be seen from (4.25).) The following proposition, which may have a number of variants, can be regarded essentially as a reformulation of this fact in the language of GGC's and the corresponding CBCI processes.

**Proposition 5.2** *Suppose that probability measures  $m_1$  and  $m_2$  on  $(0, \infty)$  are Thorin measures. For each  $i \in \{1, 2\}$ , let the quadruplet  $(1, b_i, n_i, 1)$  is the one determined from the pair  $(0, m_i)$  by Theorem 4.3. Then the following assertions hold.*

(i)  $m_1 \uplus m_2$  is a Thorin measure and the GGC with pair  $(0, m_1 \uplus m_2)$  is a unique stationary distribution of the CBCI-process with quadruplet  $(1, b_1 + b_2, n_1 + n_2, 1)$ .

(ii) For each  $t > 0$ ,  $m_1^{\uplus t}$  is a Thorin measure and the GGC with pair  $(0, m_1^{\uplus t})$  is a unique stationary distribution of the CBCI-process with quadruplet  $(1, tb_1, tn_1, 1)$ .

*Proof.* (i) According to (4.10),  $n_i$  are of the form  $n_i(dy) = dy \int u^2 e^{-uy} M_i(dy)$  for some  $M_i \in \mathcal{M}$ . Observe from (4.5) with  $a = 1$  that

$$\left( \int \frac{m_i(du)}{\lambda + u} \right)^{-1} - \lambda = b_i + \lambda \int \frac{M_i(du)}{\lambda + u} \geq 0, \quad \lambda > 0. \quad (5.18)$$

Combining the above inequality with (5.16), we get

$$\left( \int \frac{(m_1 \uplus m_2)(du)}{\lambda + u} \right)^{-1} - \lambda \geq \left( \int \frac{m_1(du)}{\lambda + u} \right)^{-1} - \lambda, \quad \lambda > 0$$

or  $\int (\lambda + u)^{-1} (m_1 \uplus m_2)(du) \leq \int (\lambda + u)^{-1} m_1(du)$  for  $\lambda > 0$ . By integrating with respect to the Lebesgue measure  $d\lambda$  over  $[0, 1]$  and then applying Fubini's theorem  $\int \log(1 + u^{-1}) (m_1 \uplus m_2)(du) \leq \int \log(1 + u^{-1}) m_1(du) < \infty$ . Hence  $m_1 \uplus m_2$  is a Thorin measure. The last half of the assertion follows from the equalities for  $i = 1$  and  $i = 2$  in (5.18). Indeed, summing up them leads to

$$\left( \int \frac{(m_1 \uplus m_2)(du)}{\lambda + u} \right)^{-1} - \lambda = (b_1 + b_2) + \lambda \int \frac{(M_1 + M_2)(du)}{\lambda + u}$$

or

$$\int \frac{(m_1 \uplus m_2)(du)}{\lambda + u} = \frac{1}{\lambda + (b_1 + b_2) + \lambda \int \frac{(M_1 + M_2)(du)}{\lambda + u}}$$

for all  $\lambda > 0$ . This is sufficient for the proof of (i).

The proof of (ii) proceeds along the same lines as that of (i) on noting that by (5.17) and (5.18)

$$\int \frac{m_1^{\uplus t}(du)}{\lambda + u} = \frac{1}{\lambda + tb_1 + \lambda t \int \frac{M_1(du)}{\lambda + u}} \leq \frac{\max\{t^{-1}, 1\}}{\lambda + b_1 + \lambda \int \frac{M_1(du)}{\lambda + u}}$$

for any  $\lambda > 0$ . The details are omitted. ■

The final topic is related to the free probability theory. In that theory, the counterpart (in an appropriate sense) of the Poisson distribution exists and is called the free Poisson distribution. It is, by definition, of the form

$$P_{\alpha,\beta}(du) := \begin{cases} (1-\beta)\epsilon_0(du) + \beta p_{\alpha,\beta}(u)du & (0 \leq \beta < 1) \\ p_{\alpha,\beta}(u)du & (\beta \geq 1) \end{cases}$$

for some  $\alpha > 0$  and  $\beta \geq 0$ , where

$$p_{\alpha,\beta}(u) = \frac{1}{2\pi\alpha u} \sqrt{4\alpha^2\beta - (u - \alpha(1+\beta))^2} \mathbf{1}_{[\alpha(1-\sqrt{\beta})^2, \alpha(1+\sqrt{\beta})^2]}(u).$$

Notice that  $P_{\alpha,\beta}$  is a Thorin measure if and only if  $\beta \geq 1$ . The formula for the Stieltjes transform of this distribution is

$$G_{P_{\alpha,\beta}}(z) = \frac{z + \alpha(1-\beta) - \sqrt{(z + \alpha(1-\beta))^2 - 4\alpha z}}{2\alpha z}. \quad (5.19)$$

(See e.g. p.205 in [13].) We now remark that a class of the free Poisson distributions plays a special role in describing the fixed points of the correspondence between  $m$  and  $M$  defined through Lemma 4.1 although we don't have any interpretation in the context of CBI-processes. Here is an explicit statement.

**Proposition 5.3** *Let  $q \geq 0$  and suppose that  $m \in \mathcal{M}$  is non-zero. Assume that  $a, b \geq 0$  and  $M \in \mathcal{M}$  satisfy (4.5) with  $r = 0$ . Then  $m$  coincides with  $M$  if and only if  $q = 0$  or  $\overline{m}_0 < \infty$  and  $m$  is given by*

$$m(du) = \begin{cases} \frac{1-bq}{a+q} P_{\alpha,\beta}(du) & (\overline{m}_0 < \infty) \\ \frac{1}{\pi u} \sqrt{u - (b/2)^2} \mathbf{1}_{[(b/2)^2, \infty)}(u) du & (q = 0, \overline{m}_0 = \infty), \end{cases} \quad (5.20)$$

where  $\alpha = (1-bq)/(a+q)^2$  and  $\beta = (1+ab)/(1-bq)$ .

*Proof.* In view of (4.8), it is obvious that  $m = M$  if and only if  $G_m(z) - q = 1/(az - b - zG_m(z))$ . By solving it we can deduce

$$G_m(z) = \frac{(a+q)z - b - \sqrt{((a+q)z - b)^2 - 4(1-bq)z}}{2z}.$$

In the case where  $\overline{m}_0 < \infty$ , we have  $a+q > 0$  by (4.6), and the proof concludes by comparing this with (5.19). If  $\overline{m}_0 = \infty$  (and hence  $\overline{M}_0 = \infty$ ), then (4.7) and (4.6) imply  $q = 0$  and  $a = 0$ , respectively. Consequently, we have  $G_m(z) = (-b - \sqrt{b^2 - 4z})/(2z)$ , from which the second expression in (5.20) can be derived by inversion. ■

Denoting by  $\rho_{q,a,b}(u)$  and  $\rho_b(u)$  the densities of the measures on the right side of (5.20) in the first and the second cases, respectively, we note that

$$\lim_{a \downarrow 0} \rho_{0,a,b}(u) = \rho_b(u) = \lim_{q \downarrow 0} \rho_{q,0,b}(u)$$

for each  $u > 0$  and  $b \geq 0$ . Remark also that  $\rho_0(u)du$  coincides with  $m(du)$  in (4.16) with  $\alpha = 1/2$  and  $\kappa = 0$ , for which clearly  $m = M$  holds.

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## References

- [1] Cox, J.C., Ingersoll, Jr., J.E., Ross, S.A., A theory of the term structure of interest rates, *Econometrica* 53 (1985) 385–407.
- [2] Bondesson, L., Generalized gamma convolutions and related classes of distributions and densities, *Lecture Notes in Statistics*, 76, Springer-Verlag, New York, 1992.
- [3] Bozejko, M., Wysoczanski, J., Remarks on  $t$ -transformations of measures and convolutions, *Ann. Inst. H. Poincaré Probab. Statist.* 37 (2001) 737–761.
- [4] Duffie, D., Filipović, D., Schachermayer, W., Affine processes and applications in finance, *Ann. Appl. Probab.* 13 (2003) 984–1053.
- [5] Hougaard, P., Survival models for heterogeneous populations derived from stable distributions, *Biometrika* 73 (1986) 387–396.
- [6] James, L.F., Roynette, B., Yor, M., Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples, *Probability Surveys* 5 (2008) 346–415.
- [7] Kawazu, K., Watanabe, S., Branching processes with immigration and related limit theorems, *Theor. Probability Appl.* 16 (1971) 36–54.
- [8] Keller-Ressel, M., Mijatović, A., On the limit distributions of continuous-state branching processes with immigration, preprint, 2011, available at <http://arxiv.org/abs/1103.5605>
- [9] Komorowski, T., Olla, S., On the sector condition and homogenization of diffusions with a Gaussian drift, *J. Funct. Anal.* 197 (2003) 179–211.
- [10] Krein, M.G., Nudelman, A. A., The Markov moment problem and extremal problems, *Translations of Mathematical Monographs*, Vol. 50, American Mathematical Society, Providence, 1977.
- [11] Li, Z., *Measure-valued branching Markov processes*, Springer, Heidelberg, 2011.
- [12] Ma, Z.M., Röckner, M., *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer, Berlin, 1992.

- [13] Nica, A., Speicher, R., Lectures on the combinatorics of free probability, London Mathematical Society Lecture Note Series, 335, Cambridge University Press, Cambridge, 2006.
- [14] Ogura, Y., Spectral representation for branching processes with immigration on the real half line, *Publ. Res. Inst. Math. Sci.* 6 (1970) 307–321.
- [15] Osada, H., Saitoh, T., An invariance principle for non-symmetric Markov processes and reflecting diffusions in random domains, *Probab. Theory Related Fields* 101 (1995) 45–63.
- [16] Pinsky, M. A., Limit theorems for continuous state branching processes with immigration, *Bull. Amer. Math. Soc.* 78 (1972) 242–244.
- [17] Sato, K., Lévy processes and infinitely divisible distributions, Cambridge University Press, Cambridge, 1999.
- [18] Schilling, R. L., Song, R., Vondraček, Z., Bernstein functions. Theory and applications, de Gruyter Studies in Mathematics, 37, Walter de Gruyter & Co., Berlin, 2010.
- [19] Speicher, R., Woroudi, R., Boolean convolution, in: Free probability theory (Waterloo, ON, 1995), pp. 267–279, *Fields Inst. Commun.* 12, Amer. Math. Soc., 1997.
- [20] Stannat, W., Spectral properties for a class of continuous state branching processes with immigration, *J. Funct. Anal.* 201 (2003) 185–227.
- [21] Stannat, W., On the Poincaré inequality for infinitely divisible measures, *Potential Anal.* 23 (2005) 279–301.
- [22] Steutel, F.W., van Harn, K., Infinite divisibility of probability distributions on the real line, Marcel Dekker, Inc., New York, 2004.
- [23] Varadhan, S.R.S., Self diffusion of a tagged particle in equilibrium for asymmetric mean zero random walks with simple exclusion, *Ann. Inst. Henri Poincaré, Probab. Statist.* 31 (1995) 273–285.
- [24] Widder, D. V., The Laplace transform, Princeton University Press, Princeton, N. J., 1941.